Invariant classification of orthogonally separable Hamiltonian systems in Euclidean space

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Abstract. The problem of the invariant classification of the orthogonal coordinate webs defined in Euclidean space is solved within the framework of Felix Klein's Erlangen Program. The results are applied to the problem of integrability of the Calogero-Moser model.

1 Introduction

In his famous *Erlangen Program* [11], Felix Klein introduced a unified point of view according to which many different branches of geometry could be integrated into a single system. As is well known, this standpoint stipulates that the main goal of any branch of geometry can be formulated as follows:

> *"Given a manifold and a group of transformations of the manifold, to study the manifold configurations with respect to those features that are not altered by the transformations of the group."* ([13], p 67)

The term "manifold of n dimensions" in this setting describes a set of n variables that independently take on the real values from $-\infty$ to ∞ ([12], p 116).

Motivated by this idea, one can assert that Euclidean geometry of $\mathbb{E}^3$ (Euclidean space) can be completely characterized by the invariants of the Euclidean group of transformations. As is well-known, this Lie group of (orientation-preserving) isometries, denoted here by $I(\mathbb{E}^3)$, is a semi-direct product of the corresponding groups of rotations and translations.

An important aspect of Euclidean geometry is the theory of orthogonal coordinate webs that originated in works of a number of eminent mathematicians of the past including Stäckel [27], Bôcher [4], Darboux [5] and Eisenhart [8] within the framework of the theory of separation of variables. Its modern developments can be found in the review by Benenti [2] and the relevant references therein. In particular, it has been shown that there exist exactly eleven orthogonal coordinate webs which afford separation of variables for the Schrödinger and Hamilton-Jacobi equations defined in $\mathbb{E}^3$. These coordinate webs are confocal quadrics determined by the Killing tensors of valence two having orthogonally integrable (normal) eigenvectors and distinct eigenvalues. Eisenhart's results in $\mathbb{E}^3$ were extended by Olevsky [21] to three-dimensional spaces of non-zero constant curvature, while Kalnins, Miller and

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others generalized them to spaces of higher dimensions (see Kalnins [10] and the references therein).

This work is a natural continuation of the project initiated in [15] (see also [18] and [26]) where isometry group invariants and covariants of valence-two Killing tensors are derived and used to classify orthogonal coordinate webs of the Euclidean and Minkowski planes. Accordingly, we approach the problem of classification of the eleven orthogonal webs in $\mathbb{E}^3$ from the viewpoint of the invariant theory of the isometry group $I(\mathbb{E}^3)$. Recall that the standard approach to the study of Killing tensors defined in pseudo-Riemannian manifolds of constant curvature rests on the fact that they can be expressed in this case as sums of symmetrized tensor products of Killing vectors. In contrast to the conventional view, we consider the Killing tensors of valence two defined in $\mathbb{E}^3$ to be algebraic objects or elements of the corresponding vector space $\mathcal{K}^2(\mathbb{E}^3)$ and define the action of $I(\mathbb{E}^3)$ in this vector space to derive $I(\mathbb{E}^3)$-*invariants* of the valence-two Killing tensors.

> *In line with the postulates of the Erlangen Program, we completely solve the problem of classification of the eleven orthogonal webs in* $\mathbb{E}^3$ *by employing the* $I(\mathbb{E}^3)$*-invariants of the vector space* $\mathcal{K}^2(\mathbb{E}^3)$ *and its subspaces.*

Our solution is based on the result of theorem 5.1 which describes the space of all isometry group invariants of the vector space of Killing tensors of valence two defined in $\mathbb{E}^3$ combined with a careful study of the corresponding vector space of Killing vectors (the Lie algebra of the isometry group of $\mathbb{E}^3$).

It must be emphasized that the problem of the invariant classification of the orthogonal coordinate webs in $\mathbb{E}^3$ is significantly more complicated than the corresponding problems in two-dimensional pseudo-Riemannian spaces of constant curvature. Apart from the obvious difficulties in dealing with a vector space of a much higher dimension, one has to solve the problem of the normality of eigenvectors of valence-two Killing tensors. More specifically, the eleven orthogonal coordinate webs in $\mathbb{E}^3$ are generated by Killing tensors of valence two with normal eigenvectors. On the other hand, unlike the situation in two-dimensional spaces, in $\mathbb{E}^3$ not every Killing tensor of valence two with distinct eigenvalues has normal eigenvectors. Moreover, the normality condition is equivalent to a system of non-linear partial differential equations (PDEs), which makes it nearly impossible to verify directly. The problem of finding necessary and sufficient *intrinsic* conditions for the eigenvectors of a tensor field of valence two with pointwise distinct eigenvalues to be orthogonally integrable has a long history. It can be traced back to Schouten [24], where such conditions depending on the eigenvectors were derived. Tonolo [30] subsequently determined a set of eigenvector-independent necessary and sufficient conditions for the Ricci tensor defined in a three-dimensional space to have normal eigenvectors. This criterion was shown by Schouten to be applicable to arbitrary (Ricci or not) valence-two tensor fields defined in an arbitrary pseudo-Riemannian manifold. Later, Nijenhuis [20] derived an equivalent formulation of the criterion introduced originally by Tonolo in terms of the components of the Nijenhuis tensor of the tensor field in question. In view of their respective contributions, we refer to these remarkable formulae throughout this paper as the *Tonolo-Schouten-Nijenhuis (TSN) conditions* and employ them to verify the normality of the eigenvectors of valence-two Killing tensors defined in $\mathbb{E}^3$. In addition, we determine in each case the coordinate transformation from the given Cartesian coordinates to the corresponding coordinate system determined by the orthogonal web.

As an illustration of the power of the new theory, we use it to obtain a concise solution to the problem of integrability of the Calogero-Moser Hamiltonian system defined in $\mathbb{E}^3$ via orthogonal separation of variables in the associated Hamilton-Jacobi equation.

The paper is organized as follows. In section 2, we give an overview of the invariant theory for vector spaces of Killing tensors defined on pseudo-Riemannian manifolds of constant curvature. This theory is then specialized in section 3 to vector spaces of valence-two Killing tensors in Euclidean space. In section 4, we discuss Hamilton-Jacobi theory in the context of separation of variables and use it to derive canonical forms for the orthogonal coordinate webs of $\mathbb{E}^3$. The fundamental invariants are derived in section 5 and are used in section 6 to classify the coordinate webs. Methods for transforming a given Killing tensor to canonical form are treated in section 7. In section 8, we summarize the steps in our algorithm and apply it in section 9 to determine separable coordinates for the Calogero-Moser system in $\mathbb{E}^3$. Finally, we draw conclusions in section 10 and indicate future research directions.

The reader will no doubt realize that our classification of the coordinate webs and our algorithm for determining separable coordinates for natural Hamiltonians in $\mathbb{E}^3$ is highly computational. Nevertheless, all computations are *purely algebraic* in nature, and thus are straightforward to implement in a computer algebra system. To complement the paper, we have written a Maple package, called the `KillingTensor` package, which performs all the steps in our algorithm. The package is available through the Maple Application Centre at http://www.mapleapps.com.

2 Invariant theory of Killing tensors

In the past decade, the classical invariant theory of homogeneous polynomials has become an active field of research once again (see Olver [22] and the references therein). The theory emerged in the nineteenth century as the intrinsic study of vector spaces of homogeneous polynomials under the action of the general linear group. Two of the authors (RGM, RGS) and Dennis The have incorporated the basic ideas of classical invariant theory into the study of Killing tensors defined in pseudo-Riemannian spaces of constant curvature under the action of the isometry group [14, 15, 16, 17, 18]. This synergy of the two theories grew out of the observation that Killing tensors of the same valence defined in a space of constant curvature constitute a vector space or, more precisely, a representation space of the isometry group of the underlying space. Putting this observation in proper perspective allows one to extend the basic ideas of classical invariant theory to the study of Killing tensors. Indeed, let $(M, \boldsymbol{g})$ be an n-dimensional pseudo-Riemannian manifold of constant curvature with metric tensor $\boldsymbol{g}$.

Definition 2.1. *A* ***Killing tensor*** $\boldsymbol{K}$ ***of valence*** $\boldsymbol{p}$ *defined in* $(M, \boldsymbol{g})$ *is a symmetric* $(p, 0)$ *tensor satisfying the Killing tensor equation*

$$[\boldsymbol{K}, \boldsymbol{g}] = 0, \tag{2.1}$$

where $[,]$ *denotes the Schouten bracket [25]. When* $p = 1$, $\boldsymbol{K}$ *is said to be a* ***Killing vector*** *(infinitesimal isometry) and the equation (2.1) reads*

$$\mathcal{L}_{\boldsymbol{K}}\boldsymbol{g} = 0, \tag{2.2}$$

where $\mathcal{L}$ *denotes the Lie derivative operator.*

The Schouten bracket is a real bilinear operator, which property together with (2.1) implies that the set $\mathcal{K}^p(M)$ of all Killing tensors of valence p defined in $(M, \boldsymbol{g})$ is in fact a vector space. Its dimension d is determined by the *Delong-Takeuchi-Thompson (DTT) formula* [7, 28, 29]

$$d = \dim \mathcal{K}^p(M) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}, \quad p \geqslant 1. \tag{2.3}$$

Therefore the general element of $\mathcal{K}^p(M)$ is represented by d arbitrary parameters $a^1, \ldots, a^d$, with respect to an appropriate basis. Alternatively, this fact can be verified by solving the corresponding Killing tensor equation (2.1) with respect to a fixed system of coordinates, in which case the parameters $a^1, \ldots, a^d$ appear as constants of integration in the general form of elements of $\mathcal{K}^p(M)$.

Each element h of the isometry group $I(M)$ induces, by the push forward map, a non-singular linear transformation $\rho(h)$ of $\mathcal{K}^p(M)$. By theorem 3.5 of McLenaghan *et al* [19], the map

$$\rho : I(M) \to GL(\mathcal{K}^p(M)) \tag{2.4}$$

defines a representation of $I(M)$. Indeed, ρ is a group isomorphism. Once the form of the general element $\boldsymbol{K}$ of $\mathcal{K}^p(M)$ is available with respect to some convenient system of coordinates on M, the explicit form of the transformation $\rho(h)\boldsymbol{K}$ (written more succinctly as $h \cdot \boldsymbol{K}$) may be written explicitly in terms of the parameters $a^1, \ldots, a^d$. We shall be particularly concerned with the smooth real-valued functions on $\mathcal{K}^p(M)$ that are invariant under the group $I(M)$. The precise definition of such $I(M)$-invariant functions of $\mathcal{K}^p(M)$ is as follows.

Definition 2.2. *Let $(M, \boldsymbol{g})$ be a pseudo-Riemannian manifold of constant curvature. Let $p \geqslant 1$ be fixed. A smooth function $F : \mathcal{K}^p(M) \to \mathbb{R}$ is said to be an **$\boldsymbol{I(M)}$-invariant of $\boldsymbol{\mathcal{K}^p(M)}$** iff it satisfies the condition*

$$F(h \cdot \boldsymbol{K}) = F(\boldsymbol{K}) \tag{2.5}$$

for $\boldsymbol{K} \in \mathcal{K}^p(M)$ and for all $h \in I(M)$.

The main problem of any invariant theory is to describe the whole space of invariants of a vector space under the action of a group. To achieve this one has to determine a set of *fundamental invariants* with the property that any other invariant is an analytic function of the fundamental invariants (see [22] for more details). The fundamental theorem of invariants of a regular Lie group action [22] determines the number of fundamental invariants needed to define the whole of the space of $I(M)$-invariants.

Theorem 2.1. *Let G be a Lie group acting regularly on an n-dimensional manifold M with s-dimensional orbits. Then, in a neighbourhood N of each point $p \in M$, there exist $n-s$ functionally independent G-invariants $\Delta_1, \ldots, \Delta_{n-s}$. Any other G-invariant $\mathcal{I}$ defined near p can be locally uniquely expressed as an analytic function of the fundamental invariants through $\mathcal{I} = F(\Delta_1, \ldots, \Delta_{n-s})$.*

In order to determine the form of the invariants of $\mathcal{K}^p(M)$, we use the fact the the invariance of a function under an entire Lie group is equivalent to the invariance of the function under the infinitesimal transformations of the group given by the corresponding Lie algebra. The precise result is given in the following proposition [22].

Proposition 2.1. *Let G be a connected Lie group of transformations acting regularly on a manifold M. A smooth real-valued function $F : M \to \mathbb{R}$ is a G-invariant iff*

$$\boldsymbol{v}(F) = 0 \tag{2.6}$$

for all $p \in M$ and for every infinitesimal generator $\boldsymbol{v}$ of G.

In our application, G is the representation $\rho(I(M))$ defined by (2.4) and the condition (2.6) is equivalent to

$$\boldsymbol{U}_i(F) = 0, \quad i = 1, \dots, r, \tag{2.7}$$

where the $\boldsymbol{U}_i$ are vector fields which form a basis of the Lie algebra of the representation and $r = \dim I(M) = \frac{1}{2}n(n+1)$. By theorem 3.5 of [19], this Lie algebra is isomorphic to the Lie algebra of $I(M)$. Such a basis may be computed directly as the basis of the tangent space to $\rho(I(M))$ at the identity if an explicit form of the representation is available. According to theorem 2.1 of the present paper, the general solution of the system of first-order PDEs (2.7) is an analytic function F of a set of fundamental $I(M)$-invariants. The number of fundamental invariants is $d - s$, where d is given by (2.3) and s is the dimension of the orbits of $\rho(I(M))$ acting regularly in the space $\mathcal{K}^p(M)$.

To determine s and the subspaces of $\mathcal{K}^p(M)$ where the isometry group $I(M)$ acts with orbits of the same dimension, one can use the result of the following proposition [22].

Proposition 2.2. *Let a Lie group G act on M and let $p \in M$. The vector space $S|_p = \mathrm{span}\{\boldsymbol{U}_i|_p | \boldsymbol{U}_i \in \mathfrak{g}\}$ spanned by all vector fields determined by the infinitesimal generators at p coincides with the tangent space to the orbit $\mathcal{O}_p$ of G that passes through p, i.e. $S|_p = T_p(\mathcal{O}_p)$. In particular, the dimension of $\mathcal{O}_p$ equals the dimension of $S|_p$.*

We are now prepared to apply the theory presented thus far to the vector space $\mathcal{K}^2(\mathbb{E}^3)$.

3 Invariant theory of Killing tensors of valence two in Euclidean space

We now specialize the general theory of the previous section to the vector space $\mathcal{K}^2(\mathbb{E}^3)$ of valence-two Killing tensors in Euclidean space $\mathbb{E}^3$. Recall the following well-known result in [23] from invariant theory.

Theorem 3.1. *The orbits of a compact linear group acting in a real vector space are separated by the fundamental (polynomial) invariants.*

We first note that in our case the group is non-compact and so in order to distinguish between the orbits of $I(\mathbb{E}^3)$ acting in the vector space $\mathcal{K}^2(\mathbb{E}^3)$ we need to employ a more elaborate analysis than a mere computation of a set of fundamental invariants.

It is well-known that in $\mathbb{E}^3$, as in all manifolds of constant curvature, any Killing tensor is expressible as a sum of symmetrized products of Killing vectors. The six Killing vectors in $\mathbb{E}^3$ may be written in Cartesian coordinates x^i viz

$$\boldsymbol{X}_i = \frac{\partial}{\partial x^i}, \qquad \boldsymbol{R}_i = \epsilon^k{}_{ji} x^j \boldsymbol{X}_k, \tag{3.1}$$

for $i = 1, 2, 3$, where ϵ_{ijk} is the Levi-Civita permutation tensor[4]. We also note the commutation relations

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = 0, \quad [\boldsymbol{X}_i, \boldsymbol{R}_j] = \epsilon^k{}_{ij}\boldsymbol{X}_k, \quad [\boldsymbol{R}_i, \boldsymbol{R}_j] = \epsilon^k{}_{ij}\boldsymbol{R}_k. \tag{3.2}$$

Thus the general Killing tensor in $\mathcal{K}^2(\mathbb{E}^3)$ may be expressed as

$$\boldsymbol{K} = A^{ij}\boldsymbol{X}_i \odot \boldsymbol{X}_j + 2B^{ij}\boldsymbol{X}_i \odot \boldsymbol{R}_j + C^{ij}\boldsymbol{R}_i \odot \boldsymbol{R}_j, \tag{3.3}$$

where the coefficients A^{ij}, B^{ij} and C^{ij} are constant and satisfy the symmetry properties

$$A^{ij} = A^{(ij)}, \quad C^{ij} = C^{(ij)}. \tag{3.4}$$

It follows from (3.1) and (3.3) that the components of the general Killing tensor in $\mathcal{K}^2(\mathbb{E}^3)$ with respect to the natural basis are given by

$$K^{ij} = A^{ij} + 2\epsilon^{(i}{}_{\ell k}B^{j)k}x^\ell + \epsilon^i{}_{mk}\epsilon^j{}_{n\ell}C^{k\ell}x^m x^n. \tag{3.5}$$

For future reference, we give explicitly the six independent components of K^{ij}. Noting the symmetries (3.4), it proves convenient to set (following [3])

$$A^{ij} = \begin{pmatrix} a_1 & \alpha_3 & \alpha_2 \\ \alpha_3 & a_2 & \alpha_1 \\ \alpha_2 & \alpha_1 & a_3 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad C^{ij} = \begin{pmatrix} c_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & c_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & c_3 \end{pmatrix} \tag{3.6}$$

and $x^i = (x, y, z)$. From (3.5) we obtain

$$\begin{aligned}
K^{11} &= a_1 - 2b_{12}z + 2b_{13}y + c_2z^2 + c_3y^2 - 2\gamma_1 yz, \\
K^{22} &= a_2 - 2b_{23}x + 2b_{21}z + c_3x^2 + c_1z^2 - 2\gamma_2 zx, \\
K^{33} &= a_3 - 2b_{31}y + 2b_{32}x + c_1y^2 + c_2x^2 - 2\gamma_3 xy, \\
K^{23} &= \alpha_1 + b_{31}z - b_{21}y + (b_{22} - b_{33})x + (\gamma_3 z + \gamma_2 y - \gamma_1 x)x - c_1 yz, \\
K^{31} &= \alpha_2 + b_{12}x - b_{32}z + (b_{33} - b_{11})y + (\gamma_1 x + \gamma_3 z - \gamma_2 y)y - c_2 zx, \\
K^{12} &= \alpha_3 + b_{23}y - b_{13}x + (b_{11} - b_{22})z + (\gamma_2 y + \gamma_1 x - \gamma_3 z)z - c_3 xy.
\end{aligned} \tag{3.7}$$

According to the DTT formula (2.3), the dimension of $\mathcal{K}^2(\mathbb{E}^3)$ is twenty which appears to disagree with (3.6) which lists twenty-one parameters. To reconcile this, we observe from (3.7) that only the differences of the diagonal coefficients b_{11}, b_{22} and b_{33} are involved. Defining

$$\beta_1 = b_{22} - b_{33}, \quad \beta_2 = b_{33} - b_{11}, \quad \beta_3 = b_{11} - b_{22}, \tag{3.8}$$

yields the constraint $\beta_1 + \beta_2 + \beta_3 = 0$, thereby showing that there are twenty independent parameters. In many of the computations which follow, it turns out to be more convenient to use the three b_{ii} parameters instead of two of the three β_i. With this in mind, we (commit an abuse of notation and) let $\mathcal{K}^2(\mathbb{E}^3)$ be the space spanned by the twenty-one parameters

$$a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3, b_{11}, b_{22}, b_{33}, b_{23}, b_{31}, b_{12}, b_{32}, b_{13}, b_{21}, c_1, c_2, c_3, \gamma_1, \gamma_2, \gamma_3. \tag{3.9}$$

[4]We are using the summation convention throughout and lowering and raising indices with the Euclidean metric $g_{ij} = \mathrm{diag}(1, 1, 1)$ and its inverse g^{ij}.

We shall also refer to (3.9) as the *Killing tensor parameters.*

We now consider the transformation rules for the Killing tensor parameters. The transformation from one set of Cartesian coordinates x^i to another set $\tilde{x}^i$ is given by

$$x^i = \lambda_j{}^i \tilde{x}^j + \delta^i, \tag{3.10}$$

where $\lambda_j{}^i \in SO(3)$ and $\delta^i \in \mathbb{R}^3$. It is straightforward to show that the Killing vectors (3.1) transform according to

$$\boldsymbol{X}_i = \lambda^j{}_i \tilde{\boldsymbol{X}}_j, \quad \boldsymbol{R}_i = \lambda^j{}_i \tilde{\boldsymbol{R}}_j + \mu^j{}_i \tilde{\boldsymbol{X}}_j, \tag{3.11}$$

where

$$\mu^j{}_i = \epsilon^k{}_{\ell i} \lambda^j{}_k \delta^\ell. \tag{3.12}$$

The Killing vector transformation rules (3.11) in conjunction with (3.3) lead to

$$\begin{aligned}
\tilde{A}^{ij} &= A^{k\ell}\lambda^i{}_k\lambda^j{}_\ell + 2B^{k\ell}\lambda^{(i}{}_k\mu^{j)}{}_\ell + C^{k\ell}\mu^i{}_k\mu^j{}_\ell, \\
\tilde{B}^{ij} &= B^{k\ell}\lambda^i{}_k\lambda^j{}_\ell + C^{k\ell}\lambda^j{}_\ell\mu^i{}_k, \\
\tilde{C}^{ij} &= C^{k\ell}\lambda^i{}_k\lambda^j{}_\ell.
\end{aligned} \tag{3.13}$$

These equations give the explicit form of the representation of $I(\mathbb{E}^3)$ on $\mathcal{K}^2(\mathbb{E}^3)$ with respect to a Cartesian coordinate system on $\mathbb{E}^3$.

Equipped with these transformation rules, we can now derive the infinitesimal generators of $I(\mathbb{E}^3)$ in the representation defined by (3.13). Let $\boldsymbol{U}_m$, $m = 1, 2, 3$, denote the generators associated to the Killing vectors $\boldsymbol{X}_m$. Noting that such Killing vectors generate translations about the x^m-axis, we set $\lambda_j{}^i = \delta_j{}^i$ in (3.13) and differentiate the resulting equations with respect to δ^m to obtain

$$\left.\frac{\partial \tilde{A}^{ij}}{\partial \delta^m}\right|_{\delta^i=0} = 2\epsilon^{(i}{}_{mk}B^{j)k}, \quad \left.\frac{\partial \tilde{B}^{ij}}{\partial \delta^m}\right|_{\delta^i=0} = \epsilon^i{}_{mk}C^{jk}, \quad \left.\frac{\partial \tilde{C}^{ij}}{\partial \delta^m}\right|_{\delta^i=0} = 0.$$

The corresponding differential operators are therefore

$$\boldsymbol{U}_i = 2\epsilon^{(j}{}_{i\ell}B^{k)\ell}\frac{\partial}{\partial A^{jk}} + \epsilon^j{}_{i\ell}C^{k\ell}\frac{\partial}{\partial B^{jk}}, \tag{3.14}$$

for $i = 1, 2, 3$, where the range of summation over the derivative operators is understood to be over only those parameters listed in (3.9). Next, let $\boldsymbol{V}_m$, $m = 1, 2, 3$, denote the generators associated to the Killing vectors $\boldsymbol{R}_m$, the generators of rotations about the x^m-axis. For an infinitesimal rotation about the x^3-axis by an angle θ^3, the rotation $\lambda_j{}^i \in SO(3)$ is given by

$$\lambda_j{}^i = \begin{pmatrix} \cos\theta^3 & -\sin\theta^3 & 0 \\ \sin\theta^3 & \cos\theta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij} \quad \Rightarrow \quad \left.\frac{\mathrm{d}\lambda_j{}^i}{\mathrm{d}\theta^3}\right|_{\theta^3=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} = \epsilon_{3j}{}^i.$$

More generally, for an infinitesimal rotation about the x^m-axis,

$$\left.\frac{\mathrm{d}\lambda_j{}^i}{\mathrm{d}\theta^m}\right|_{\theta^m=0} = \epsilon_{mi}{}^j \quad \Leftrightarrow \quad \left.\frac{\mathrm{d}\lambda^{ij}}{\mathrm{d}\theta^m}\right|_{\theta^m=0} = \epsilon^i{}_{jm}.$$

It thus follows from (3.13) that

$$
\begin{aligned}
\boldsymbol{V}_i = {} & (\epsilon^j{}_{\ell i} A^{\ell k} + \epsilon^k{}_{\ell i} A^{j\ell}) \frac{\partial}{\partial A^{jk}} + (\epsilon^j{}_{\ell i} B^{\ell k} + \epsilon^k{}_{\ell i} B^{j\ell}) \frac{\partial}{\partial B^{jk}} \\
& + (\epsilon^j{}_{\ell i} C^{\ell k} + \epsilon^k{}_{\ell i} C^{j\ell}) \frac{\partial}{\partial C^{jk}},
\end{aligned}
\tag{3.15}
$$

for $i = 1, 2, 3$. As required by the general theory, the generators (3.14) and (3.15) satisfy the same commutation relations as the Killing vectors (3.1), namely

$$
[\boldsymbol{U}_i, \boldsymbol{U}_j] = 0, \quad [\boldsymbol{U}_i, \boldsymbol{V}_j] = \epsilon^k{}_{ij} \boldsymbol{U}_k, \quad [\boldsymbol{V}_i, \boldsymbol{V}_j] = \epsilon^k{}_{ij} \boldsymbol{V}_k.
$$

For computational purposes, we shall require explicit expressions for the generators. It follows from (3.14) and (3.15) that

$$
\boldsymbol{U}_i = \sum_{j=1}^{21} \mathcal{G}_i{}^j \frac{\partial}{\partial a^j}, \quad \boldsymbol{V}_i = \sum_{j=1}^{21} \mathcal{G}_{i+3}{}^j \frac{\partial}{\partial a_j},
\tag{3.16}
$$

for $i = 1, 2, 3$, where a^j, $j = 1, \ldots, 21$, are the twenty-one Killing tensor parameters ordered by (3.9) and

$$
\mathcal{G}_i{}^j = \left(\begin{array}{cccccccc}
0 & -2b_{23} & 2b_{32} & b_{22} - b_{33} & b_{12} & -b_{13} & 0 & -\gamma_1 \\
2b_{13} & 0 & -2b_{31} & -b_{21} & b_{33} - b_{11} & b_{23} & \gamma_2 & 0 \\
-2b_{12} & 2b_{21} & 0 & b_{31} & -b_{32} & b_{11} - b_{22} & -\gamma_3 & \gamma_3 \\
0 & 2\alpha_1 & -2\alpha_1 & a_3 - a_2 & -\alpha_3 & \alpha_2 & 0 & b_{23} + b_{32} \\
-2\alpha_2 & 0 & 2\alpha_2 & \alpha_3 & a_1 - a_3 & -\alpha_1 & -b_{31} - b_{13} & 0 \\
2\alpha_3 & -2\alpha_3 & 0 & -\alpha_2 & \alpha_1 & a_2 - a_1 & b_{12} + b_{21} & -b_{12} - b_{21}
\end{array} \right.
$$

$$
\begin{array}{ccccccc}
\gamma_1 & -c_3 & \gamma_3 & 0 & c_2 & 0 & -\gamma_2 \\
-\gamma_2 & 0 & -c_1 & \gamma_1 & -\gamma_3 & c_3 & 0 \\
0 & \gamma_2 & 0 & -c_2 & 0 & -\gamma_1 & c_1 \\
-b_{23} - b_{32} & b_{33} - b_{22} & -b_{21} & b_{13} & b_{33} - b_{22} & -b_{12} & b_{31} \\
b_{31} + b_{13} & b_{21} & b_{11} - b_{33} & -b_{32} & b_{12} & b_{11} - b_{33} & -b_{23} \\
0 & -b_{13} & b_{32} & b_{22} - b_{11} & -b_{31} & b_{23} & b_{22} - b_{11}
\end{array}
$$

$$
\left. \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\gamma_1 & -2\gamma_1 & c_3 - c_2 & -\gamma_3 & \gamma_2 \\
-2\gamma_2 & 0 & 2\gamma_2 & \gamma_3 & c_1 - c_3 & -\gamma_1 \\
2\gamma_3 & -2\gamma_3 & 0 & -\gamma_2 & \gamma_1 & c_2 - c_1
\end{array} \right)_{ij} .
$$

Finally, we observe that the coefficient matrix $\mathcal{G}_i{}^j$ has rank six almost everywhere, and so, in view of theorem 2.1, we expect fifteen fundamental $I(\mathbb{E}^3)$-invariants. The computation and presentation of these invariants are treated in section 5.

4 Hamilton-Jacobi theory and orthogonal coordinate webs

Consider a Hamiltonian system defined on $(M, \boldsymbol{g})$ by a natural Hamiltonian function of the form

$$
H = \tfrac{1}{2} g^{ij}(\boldsymbol{x}) p_i p_j + V(\boldsymbol{x}), \quad i, j = 1, \ldots n,
\tag{4.1}
$$

with respect to the canonical Poisson bi-vector $\boldsymbol{P} = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p^i}$ given in terms of the position-momenta coordinates $(\boldsymbol{x}, \boldsymbol{p}) = (x^i, p_i)$, $i = 1, \ldots, n$ on the cotangent bundle $T^*(M)$. As is well-known, in many cases the Hamiltonian system defined by (4.1) can be integrated by quadratures by finding a complete integral W of the corresponding Hamilton-Jacobi equation which is a first-order PDE given by

$$\frac{1}{2} g^{ij}(\boldsymbol{x}) \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} + V(\boldsymbol{x}) = E, \quad p_j = \frac{\partial W}{\partial x^i}. \tag{4.2}$$

The geometrical meaning of equation (4.2) and its complete integral W is well-understood (see, for example, [1]). Thus, if the function $F = \frac{1}{2} g^{ij} W_{,i} W_{,j} + V - E = 0$ is regular on the cotangent bundle $T^*(M)$, then equation (4.2) defines a hypersurface in $T^*(M)$. Furthermore, W is a complete integral of (4.2) iff the Lagrangian submanifold $\mathcal{S} \subset T^*(M)$ determined by the equations $p_i = W_{,i}$ lies on the hypersurface defined by the Hamilton-Jacobi equation (4.2). Solving (4.2) is normally based on finding a canonical transformation to separable coordinates: $(\boldsymbol{x}, \boldsymbol{p}) \rightarrow (\boldsymbol{u}, \boldsymbol{v})$ with respect to which the equation can be solved under the additive separation ansatz $W(\boldsymbol{u}; \boldsymbol{c}) = \sum_{i=1}^{n} W_i(u^i; \boldsymbol{c})$ and the non-degeneracy condition $\det(\partial^2 W / \partial u^i \partial c_j)_{n \times n} \neq 0$, where $\boldsymbol{c} = (c_1, \ldots, c_n)$ is a constant vector. Orthogonal separation of variables occurs in the case when the transformations to separable coordinates are point-transformations and the metric tensor $\boldsymbol{g}$ is diagonal with respect to the coordinates of separation $(\boldsymbol{u}, \boldsymbol{v})$. A useful criterion for orthogonal separability is given by Benenti [2].

Theorem 4.1. *The Hamiltonian system defined by (4.1) is orthogonally separable if and only if there exists a valence-two Killing tensor* $\boldsymbol{K}$ *with (i) pointwise simple and real eigenvalues, (ii) orthogonally integrable (normal) eigenvectors and (iii) such that*

$$\mathrm{d}(\boldsymbol{K}\, \mathrm{d}V) = 0. \tag{4.3}$$

A Killing tensor satisfying conditions (i) and (ii) of theorem 4.1 is called a *characteristic Killing tensor (CKT).*

Let us elaborate on conditions (i) and (ii) in theorem 4.1. On two-dimensional Riemannian manifolds of constant curvature these conditions are trivial, since every eigenvector $\boldsymbol{\xi}$ of $\boldsymbol{K}$ is normal and $\boldsymbol{K}$ has repeated eigenvalues iff it is a multiple of the metric $\boldsymbol{g}$. In three-dimensions, $\mathbb{E}^3$ in particular, the situation is far more complicated. Computing eigenvectors of a symmetric 3×3 tensor is tedious and becomes virtually intractable if one considers Killing tensors with arbitrary parameters. Instead, we employ the Tonolo-Schouten-Nijenhuis (TSN) conditions, as introduced in section 1, which are both necessary and sufficient for a given symmetric (Killing) tensor field to have integrable eigenvectors. These conditions read

$$N^{\ell}{}_{[jk} g_{i]\ell} = 0, \tag{4.4a}$$

$$N^{\ell}{}_{[jk} K_{i]\ell} = 0, \tag{4.4b}$$

$$N^{\ell}{}_{[jk} K_{i]m} K^{m}{}_{\ell} = 0 \tag{4.4c}$$

where $N^i{}_{jk}$ are the components of the Nijenhuis tensor of K^{ij} given by

$$N^i{}_{jk} = K^i{}_{\ell} K^{\ell}{}_{[j,k]} + K^{\ell}{}_{[j} K^i{}_{k],\ell}. \tag{4.5}$$

We remark that the TSN conditions (4.4a)–(4.4c) yield 10 quadratic, 35 cubic and 84 quartic equations, respectively, in the Killing tensor parameters. It is thus straightforward to verify, using (4.4), if a given Killing tensor satisfies condition (ii) of theorem 4.1. However, we have been unable to solve the conditions (4.4) directly to obtain the most general Killing tensor admitting orthogonally integrable eigenvectors[5]. Condition (i), namely the distinct eigenvalues condition, like condition (ii), can also be verified directly for a given Killing tensor: we simply compute the discriminant of the characteristic polynomial of K^{ij} and verify that it does not vanish identically.

Although a general solution of the TSN conditions (4.4) appears intractable, we may instead employ Eisenhart's method [8] to derive all Killing tensors with normal eigenvectors up to equivalence. In particular, each representative Killing tensor characterizes separability of the Hamilton-Jacobi equation (4.2) in one of the eleven (orthogonally) separable coordinate systems in $\mathbb{E}^3$. The Eisenhart method can be described as follows. Consider the Euclidean metric

$$\mathrm{d}s^2 = g_{11}(\mathrm{d}u^1)^2 + g_{22}(\mathrm{d}u^2)^2 + g_{33}(\mathrm{d}u^3)^2,$$

with respect to separable coordinates u^i. The method yields three canonical Killing tensors given by

$$K_{ij} = \mathrm{diag}(\lambda_1\, g_{11}, \lambda_2\, g_{22}, \lambda_3\, g_{33}),$$

where the λ_i satisfy the linear system of PDEs

$$\frac{\partial \lambda_i}{\partial u^j} = (\lambda_i - \lambda_j)\frac{\partial}{\partial u^j} \ln g_{ii}, \quad \text{(no sum)} \tag{4.6}$$

for $i, j = 1, 2, 3$ (see [8], equations (1.8)). Trivially, any multiple of the metric $\boldsymbol{g}$ satisfies (4.6). It is straightforward to solve (4.6) for each of the eleven separable coordinate systems in $\mathbb{E}^3$ to obtain the two additional (non-trivial) canonical Killing tensors which we shall label $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$.

We now summarize the results of this calculation. For each of the eleven separable coordinate systems, we give the corresponding coordinate transformation, ranges of the separable coordinates, the metric and the components of the canonical Killing tensors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$.

$$\begin{array}{c} \text{Cartesian:} \\ (x, y, z) \\ (4.7) \end{array} \quad \left\{ \begin{array}{l} x = x,\ y = y,\ z = z \\ -\infty < x, y, z < \infty \\ \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(0, 1, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{array} \right.$$

$$\begin{array}{c} \text{Circular cylindrical:} \\ (r, \theta, z) \\ (4.8) \end{array} \quad \left\{ \begin{array}{l} x = r\cos\theta,\ y = r\sin\theta,\ z = z \\ r \geqslant 0, \quad 0 \leqslant \theta < 2\pi, \quad -\infty < z < \infty \\ \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(0, r^4, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{array} \right.$$

[5]Steve Czapor (private communication) has simplified the situation considerably. Using Gröbner basis theory, he has shown that (4.4a) and (4.4b) imply (4.4c), for any Killing tensor $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^3)$.

Parabolic cylindrical: (μ, ν, z) (4.9)

$$\begin{cases} x = \frac{1}{2}(\mu^2 - \nu^2),\ y = \mu\nu,\ z = z \\ \mu \geqslant 0, \quad -\infty < \nu < \infty, \quad -\infty < z < \infty \\ \mathrm{d}s^2 = (\mu^2 + \nu^2)(\mathrm{d}\mu^2 + \mathrm{d}\nu^2) + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(\nu^2 g_{11}, -\mu^2 g_{22}, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{cases}$$

Elliptic-hyperbolic: (η, ψ, z) (4.10)

$$\begin{cases} x = a\cosh\eta\cos\psi,\ y = a\sinh\eta\sin\psi,\ z = z \\ \eta \geqslant 0, \quad 0 \leqslant \psi < 2\pi, \quad -\infty < z < \infty, \quad a > 0 \\ \mathrm{d}s^2 = a^2(\cosh^2\eta - \cos^2\psi)(\mathrm{d}\eta^2 + \mathrm{d}\psi^2) + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(a^2\cos^2\psi\, g_{11}, a^2\cosh^2\eta\, g_{22}, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{cases}$$

Spherical: (r, θ, ϕ) (4.11)

$$\begin{cases} x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \\ r \geqslant 0, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \phi < 2\pi \\ \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \\ K_1^{ij} = \mathrm{diag}(0, r^4, r^4\sin^2\theta) \\ K_2^{ij} = \mathrm{diag}(0, 0, r^4\sin^4\theta) \end{cases}$$

Prolate spheroidal: (η, θ, ψ) (4.12)

$$\begin{cases} x = a\sinh\eta\sin\theta\cos\psi,\ y = a\sinh\eta\sin\theta\sin\psi,\ z = a\cosh\eta\cos\theta \\ \eta \geqslant 0, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \psi < 2\pi, \quad a > 0 \\ \mathrm{d}s^2 = a^2(\sinh^2\eta + \sin^2\theta)(\mathrm{d}\eta^2 + \mathrm{d}\theta^2) + a^2\sinh^2\eta\sin^2\theta\,\mathrm{d}\psi^2 \\ K_1^{ij} = \mathrm{diag}\big(-a^2\sin^2\theta\, g_{11}, a^2\sinh^2\eta\, g_{22}, a^2(\sinh^2\eta - \sin^2\theta)g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(0, 0, a^2\sinh^2\eta\sin^2\theta\, g_{33}) \end{cases}$$

Oblate spheroidal: (η, θ, ψ) (4.13)

$$\begin{cases} x = a\cosh\eta\sin\theta\cos\psi,\ y = a\cosh\eta\sin\theta\sin\psi,\ z = a\sinh\eta\cos\theta \\ \eta \geqslant 0, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \psi < 2\pi, \quad a > 0 \\ \mathrm{d}s^2 = a^2(\cosh^2\eta - \sin^2\theta)(\mathrm{d}\eta^2 + \mathrm{d}\theta^2) + a^2\cosh^2\eta\sin^2\theta\,\mathrm{d}\psi^2 \\ K_1^{ij} = \mathrm{diag}\big(a^2\sin^2\theta\, g_{11}, a^2\cosh^2\eta\, g_{22}, a^2(\cosh^2\eta + \sin^2\theta)g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(0, 0, a^2\cosh^2\eta\sin^2\theta\, g_{33}) \end{cases}$$

Parabolic: (μ, ν, ψ) (4.14)

$$\begin{cases} x = \mu\nu\cos\psi,\ y = \mu\nu\sin\psi,\ z = \frac{1}{2}(\mu^2 - \nu^2) \\ \mu \geqslant 0, \quad \nu \geqslant 0, \quad 0 \leqslant \psi < 2\pi \\ \mathrm{d}s^2 = (\mu^2 + \nu^2)(\mathrm{d}\mu^2 + \mathrm{d}\nu^2) + \mu^2\nu^2\,\mathrm{d}\psi^2 \\ K_1^{ij} = \mathrm{diag}\big(-\nu^2 g_{11}, \mu^2 g_{22}, (\mu^2 - \nu^2)g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(0, 0, \mu^2\nu^2 g_{33}) \end{cases}$$

Conical: (r,θ,λ) (4.15)

$$\begin{cases} x^2 = \left(\dfrac{r\theta\lambda}{bc}\right)^2,\ y^2 = \dfrac{r^2(\theta^2-b^2)(b^2-\lambda^2)}{b^2(c^2-b^2)},\ z^2 = \dfrac{r^2(c^2-\theta^2)(c^2-\lambda^2)}{b^2(c^2-b^2)} \\ r \geqslant 0, \quad b^2 < \theta^2 < c^2, \quad 0 < \lambda^2 < b^2, \\ \mathrm{d}s^2 = \mathrm{d}r^2 + \dfrac{r^2(\theta^2-\lambda^2)}{(\theta^2-b^2)(c^2-\theta^2)}\,\mathrm{d}\theta^2 + \dfrac{r^2(\theta^2-\lambda^2)}{(b^2-\lambda^2)(c^2-\lambda^2)}\,\mathrm{d}\lambda^2 \\ K_1^{ij} = \mathrm{diag}(0, r^2\lambda^2 g_{22}, r^2\theta^2 g_{33}) \\ K_2^{ij} = \mathrm{diag}(0, r^2 g_{22}, r^2 g_{33}) \end{cases}$$

Paraboloidal: (μ,ν,λ) (4.16)

$$\begin{cases} x^2 = \dfrac{4(\mu-b)(b-\nu)(b-\lambda)}{b-c},\ y^2 = \dfrac{4(\mu-c)(c-\nu)(\lambda-c)}{b-c}, \\ \qquad z = \mu+\nu+\lambda-b-c \\ 0 < \nu < c < \lambda < b < \mu < \infty \\ \mathrm{d}s^2 = \dfrac{(\mu-\nu)(\mu-\lambda)}{(\mu-b)(\mu-c)}\,\mathrm{d}\mu^2 + \dfrac{(\mu-\nu)(\lambda-\nu)}{(b-\nu)(c-\nu)}\,\mathrm{d}\nu^2 \\ \qquad + \dfrac{(\lambda-\nu)(\mu-\lambda)}{(b-\lambda)(\lambda-c)}\,\mathrm{d}\lambda^2 \\ K_1^{ij} = \mathrm{diag}\big(2(\nu+\lambda)g_{11}, 2(\lambda+\mu)g_{22}, 2(\mu+\nu)g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(-4\nu\lambda g_{11}, -4\lambda\mu g_{22}, -4\mu\nu g_{33}) \end{cases}$$

Ellipsoidal: (η,θ,λ) (4.17)

$$\begin{cases} x^2 = \dfrac{(a-\eta)(a-\theta)(a-\lambda)}{(a-b)(a-c)},\ y^2 = \dfrac{(b-\eta)(b-\theta)(b-\lambda)}{(b-a)(b-c)}, \\ \qquad z^2 = \dfrac{(c-\eta)(c-\theta)(c-\lambda)}{(c-a)(c-b)} \\ a > \eta > b > \theta > c > \lambda \\ \mathrm{d}s^2 = \dfrac{(\eta-\theta)(\eta-\lambda)}{4(a-\eta)(b-\eta)(c-\eta)}\,\mathrm{d}\eta^2 + \dfrac{(\theta-\eta)(\theta-\lambda)}{4(a-\theta)(b-\theta)(c-\theta)}\,\mathrm{d}\theta^2 \\ \qquad + \dfrac{(\lambda-\eta)(\lambda-\theta)}{4(a-\lambda)(b-\lambda)(c-\lambda)}\,\mathrm{d}\lambda^2 \\ K_1^{ij} = \mathrm{diag}\big(-(\theta+\lambda)g_{11}, -(\lambda+\eta)g_{22}, -(\eta+\theta)g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(\theta\lambda g_{11}, \lambda\eta g_{22}, \eta\theta g_{33}) \end{cases}$$

As we are dealing with Hamiltonian systems defined in terms of *Cartesian* coordinates, the next step is to transform the components of each of the canonical Killing tensors K_1^{ij} and K_2^{ij} to Cartesian coordinates. This again is a routine calculation using the transformations from Cartesian to separable coordinates listed in (4.7)–(4.17) and the appropriate tensor transformation law. For each of the eleven separable cases, we take a linear combination of K_1^{ij}, K_2^{ij} and the metric g^{ij}. Using (3.7), we then identify the constants in the linear combination and any essential parameters[6] with the Killing tensor parameters (3.9) appearing in (3.7). We note that if the separable case under consideration has n essential parameters, then one can generally choose $n+3$ of the Killing tensor parameters in the identification. However, in the paraboloidal and ellipsoidal cases, it is convenient to

[6]These refer to any parameters appearing in the canonical Killing tensors K_1^{ij} and K_2^{ij}.

choose more than $n+3$ parameters in the identification so that the components of the resulting Killing tensor are polynomials in the parameters and the Cartesian coordinates. Consequently, this leads to algebraic constraints in the Killing tensor parameters. These constraints not only ensure that the resulting Killing tensor has normal eigenvectors, but also guarantees that one can always (uniquely) recover the original constants in the linear combination and all essential parameters from the identified Killing tensor parameters. Each of the CKTs constructed in this manner uniquely defines one of the eleven possible *orthogonal coordinate webs* in $\mathbb{E}^3$. We now present the results of this procedure. For each of the eleven separable coordinate systems, we give the components of the corresponding CKT with respect to Cartesian coordinates and any restrictions on the Killing tensor parameters.

1. Cartesian web

$$K^{ij} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{4.18}$$

2. Circular cylindrical web

$$K^{ij} = \begin{pmatrix} a_1 + c_3 y^2 & -c_3 xy & 0 \\ -c_3 xy & a_1 + c_3 x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{4.19}$$

3. Parabolic cylindrical web

$$K^{ij} = \begin{pmatrix} a_1 & b_{23} y & 0 \\ b_{23} y & a_1 - 2b_{23} x & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{4.20}$$

4. Elliptic-hyperbolic web

$$K^{ij} = \begin{pmatrix} a_1 + c_3 y^2 & -c_3 xy & 0 \\ -c_3 xy & a_2 + c_3 x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad \frac{a_1 - a_2}{c_3} > 0 \tag{4.21}$$

5. Spherical web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 xz \\ -c_3 xy & a_1 + c_3 x^2 + c_2 z^2 & -c_2 yz \\ -c_2 xz & -c_2 yz & a_1 + c_2 x^2 + c_2 y^2 \end{pmatrix} \tag{4.22}$$

6. Prolate spheroidal web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 xz \\ -c_3 xy & a_1 + c_3 x^2 + c_2 z^2 & -c_2 yz \\ -c_2 xz & -c_2 yz & a_3 + c_2 x^2 + c_2 y^2 \end{pmatrix}, \quad \frac{a_3 - a_1}{c_2} > 0 \tag{4.23}$$

7. Oblate spheroidal web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 xz \\ -c_3 xy & a_1 + c_3 x^2 + c_2 z^2 & -c_2 yz \\ -c_2 xz & -c_2 yz & a_3 + c_2 x^2 + c_2 y^2 \end{pmatrix}, \quad \frac{a_3 - a_1}{c_2} < 0 \tag{4.24}$$

8. Parabolic web

$$K^{ij} = \begin{pmatrix} a_1 - 2b_{12}z + c_3y^2 & -c_3xy & b_{12}x \\ -c_3xy & a_1 - 2b_{12}z + c_3x^2 & b_{12}y \\ b_{12}x & b_{12}y & a_1 \end{pmatrix} \tag{4.25}$$

9. Conical web

$$K^{ij} = \begin{pmatrix} a_1 + c_2z^2 + c_3y^2 & -c_3xy & -c_2zx \\ -c_3xy & a_1 + c_3x^2 + c_1z^2 & -c_1yz \\ -c_2zx & -c_1yz & a_1 + c_1y^2 + c_2x^2 \end{pmatrix} \tag{4.26}$$

10. Paraboloidal web

$$K^{ij} = \begin{pmatrix} a_1 - 2b_{12}z + c_3y^2 & -c_3xy & b_{12}x \\ -c_3xy & a_2 + 2b_{21}z + c_3x^2 & -b_{21}y \\ b_{12}x & -b_{21}y & a_3 \end{pmatrix}, \tag{4.27}$$
$$b_{12}[b_{12}b_{21} + c_3(a_2 - a_3)] + b_{21}[b_{12}b_{21} + c_3(a_1 - a_3)] = 0$$

11. Ellipsoidal web

$$K^{ij} = \begin{pmatrix} a_1 + c_2z^2 + c_3y^2 & -c_3xy & -c_2zx \\ -c_3xy & a_2 + c_3x^2 + c_1z^2 & -c_1yz \\ -c_2zx & -c_1yz & a_3 + c_1y^2 + c_2x^2 \end{pmatrix}, \tag{4.28}$$
$$(a_1 - a_2)c_1c_2 + (a_2 - a_3)c_2c_3 + (a_3 - a_1)c_3c_1 = 0$$

We remark that the eleven Killing tensors (4.18)–(4.28) represent all possible Killing tensors with normal eigenvectors, up to equivalence. We have essentially computed the general solution of the TSN conditions (4.4) using Eisenhart's method.

As we shall see in section 6, the fundamental invariants of $\mathcal{K}^2(\mathbb{E}^3)$ fail to discriminate amongst the eleven coordinate webs. In anticipation, we make the following key observation. The eleven orthogonal coordinate webs can be divided into three groups according to table 1. More precisely, we say that a CKT $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^3)$ is *translational (rotational)* if it admits a translational (rotational) Killing vector[7] $\boldsymbol{V} \in \mathcal{K}^1(\mathbb{E}^3)$ satisfying $\mathcal{L}_{\boldsymbol{V}}\boldsymbol{K} = 0$. This definition still lacks complete precision for we have not defined translational and rotational Killing vectors. Certainly, one can give a definition of such Killing vectors in terms of integral curves. But ideally we would like to give a definition in terms of algebraic invariants. We will revisit this problem in section 6.1. For now, it suffices to note that the canonical translational CKTs (4.18)–(4.21) admit the Killing vector $\boldsymbol{V} = \boldsymbol{X}_3$, while the canonical rotational CKTs (4.22)–(4.25) admit the Killing vector $\boldsymbol{V} = \boldsymbol{R}_3$.

We now pose the following problem: construct a subspace of $\mathcal{K}^2(\mathbb{E}^3)$ consisting of translational Killing tensors, say with $\boldsymbol{V} = \boldsymbol{X}_3$. To proceed we take the general Killing tensor from (3.3) and impose the condition $\mathcal{L}_{\boldsymbol{X}_3}\boldsymbol{K} = 0$. This gives a linear

[7]The circular cylindrical tensor (4.19) also admits a rotational Killing vector and can therefore be considered as both translational and rotational.

Table 1: The orthogonal coordinate webs in Euclidean space.

Translational webs	Rotational webs	Asymmetric webs
Cartesian	spherical	conical
circular cylindrical	prolate spheroidal	paraboloidal
parabolic cylindrical	oblate spheroidal	ellipsoidal
elliptic-hyperbolic	parabolic	

system of equations in the Killing tensor parameters which can be readily solved to yield

$$K^{ij} = \begin{pmatrix} a_1 + 2b_{13}y + c_3y^2 & \alpha_3 - b_{13}x + b_{23}y - c_3xy & \alpha_2 - \beta_1 y \\ \alpha_3 - b_{13}x + b_{23}y - c_3xy & a_2 - 2b_{23}x + c_3x^2 & \alpha_1 + \beta_1 x \\ \alpha_2 - \beta_1 y & \alpha_1 + \beta_1 x & a_3 \end{pmatrix}. \tag{4.29}$$

We see that the four canonical translational CKTs are all special cases of (4.29). However, it follows that the Killing tensor (4.29) does not generally have normal eigenvectors. Consequently, we cannot take our subspace to be those Killing tensors of the form (4.29). Nevertheless, using the TSN conditions (4.4), it can be shown that $\alpha_1 = \alpha_2 = \beta_1 = 0$ is a sufficient condition for (4.29) to have normal eigenvectors. This is not a necessary condition, for the constant Killing tensor $K^{ij} = A^{ij}$ is of the form (4.29) and has normal eigenvectors. But, it is immediate from the transformation rules (3.13) that a CKT is Cartesian if and only if it is a constant Killing tensor. Moreover, it can be shown from (4.4) that if a Killing tensor of the form (4.29) has normal eigenvectors and is *not* Cartesian, then $\alpha_1 = \alpha_2 = \beta_1 = 0$. Therefore, any translational Killing tensor with orthogonally integrable eigenvectors which is not Cartesian has the form

$$K_T^{ij} = \begin{pmatrix} a_1 + 2b_{13}y + c_3y^2 & \alpha_3 - b_{13}x + b_{23}y - c_3xy & 0 \\ \alpha_3 - b_{13}x + b_{23}y - c_3xy & a_2 - 2b_{23}x + c_3x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}. \tag{4.30}$$

We define the subspace $\mathcal{K}_T^2(\mathbb{E}^3)$ of $\mathcal{K}^2(\mathbb{E}^3)$ to be the set of all Killing tensors of the form (4.30). We shall refer to this subspace as the *space of translational Killing tensors*, bearing in mind that it does not include all of the Cartesian CKTs. This does not pose any obstacle whatsoever in our goal of constructing a classification scheme, as we pointed out that Cartesian CKTs are trivially the constant Killing tensors. Finally, we remark that the space $\mathcal{K}_T^2(\mathbb{E}^3)$ enjoys two nice features. Firstly, the form of the general translational Killing tensor (4.30) is invariant under the (orientation-preserving) isometry group $I(\mathbb{E}^2)$, and secondly, the upper 2×2 block of (4.30) is the general Killing tensor on the Euclidean plane $\mathcal{K}^2(\mathbb{E}^2)$ (see, for example, [15]). Consequently, we can take advantage of known results in the literature for classifying the translational webs.

We can perform a similar analysis for the rotational webs. It follows that any Killing tensor with normal eigenvectors admitting a Killing vector $\boldsymbol{V} = \boldsymbol{R}_3$ has the form

$$K_R^{ij} = \begin{pmatrix} a_1 - 2b_{12}z + c_2z^2 + c_3y^2 & -c_3xy & b_{12}x - c_2xz \\ -c_3xy & a_1 - 2b_{12}z + c_3x^2 + c_2z^2 & b_{12}y - c_2yz \\ b_{12}x - c_2xz & b_{12}y - c_2yz & a_3 + c_2x^2 + c_2y^2 \end{pmatrix}. \tag{4.31}$$

We define the subspace $\mathcal{K}^2_R(\mathbb{E}^3)$ of $\mathcal{K}^2(\mathbb{E}^3)$ to be the set of all Killing tensors of the form (4.31) and shall refer to this subspace as the *space of rotational Killing tensors*. We remark that the form of the general rotational Killing tensor (4.31) is also invariant under the isometry group $I(\mathbb{R})$ (i.e. the group of translations about the z-axis) and that all canonical rotational CKTs (4.22)–(4.25) are special cases of (4.31).

In conclusion, we have defined the following four vector spaces: $\mathcal{K}^1(\mathbb{E}^3)$, $\mathcal{K}^2_T(\mathbb{E}^3)$, $\mathcal{K}^2_R(\mathbb{E}^3)$ and $\mathcal{K}^2(\mathbb{E}^3)$. For each of these four spaces, we need to compute the fundamental invariants under the action of the appropriate isometry group and classify the corresponding canonical forms. This is the topic of the next two sections.

5 Fundamental invariants of Killing tensors in Euclidean space

In this section we derive the fundamental invariants in each of the four vector spaces $\mathcal{K}^1(\mathbb{E}^3)$, $\mathcal{K}^2_T(\mathbb{E}^3)$, $\mathcal{K}^2_R(\mathbb{E}^3)$ and $\mathcal{K}^2(\mathbb{E}^3)$, under the action of their corresponding isometry group.

5.1 The space of Killing vectors

The most general Killing vector $\boldsymbol{V} \in \mathcal{K}^1(\mathbb{E}^3)$ may be expressed as

$$\boldsymbol{V} = A^i \boldsymbol{X}_i + C^i \boldsymbol{R}_i, \tag{5.1}$$

where $\boldsymbol{X}_i$ and $\boldsymbol{R}_i$, $i = 1, 2, 3$, are the Killing vectors defined in (3.1) and the coefficients A^i and C^i are constant. For sake of convenience, we set

$$A^i = (a_1, a_2, a_3), \quad C^i = (c_1, c_2, c_3),$$

so that the space of *Killing vector parameters* is spanned by the six parameters

$$a_1, a_2, a_3, c_1, c_2, c_3. \tag{5.2}$$

For future reference, we note that (5.1) can be written in the form

$$\boldsymbol{V} = (a_1 - c_2 z + c_3 y)\boldsymbol{X}_1 + (a_2 - c_3 x + c_1 z)\boldsymbol{X}_2 + (a_3 - c_1 y + c_2 x)\boldsymbol{X}_3. \tag{5.3}$$

As in section 3, we can derive the transformation rules for the Killing vector parameters under the action of $I(\mathbb{E}^3)$. It follows that

$$\tilde{A}^i = A^j \lambda^i{}_j + C^j \mu^i{}_j, \quad \tilde{C}^i = C^j \lambda^i{}_j, \tag{5.4}$$

which lead to the following infinitesimal generators of $I(\mathbb{E}^3)$ in the representation defined by (5.4):

$$\boldsymbol{U}_i = \epsilon^j{}_{ik} C^k \frac{\partial}{\partial A^j}, \quad \boldsymbol{V}_i = \epsilon^j{}_{ki} A^k \frac{\partial}{\partial A^j} + \epsilon^j{}_{ki} C^k \frac{\partial}{\partial C^j}, \tag{5.5}$$

for $i = 1, 2, 3$. Using the formalism of section 2, it follows from (5.5) that $\mathcal{K}^1(\mathbb{E}^3)$ admits two fundamental $I(\mathbb{E}^3)$-invariants, namely

$$\Delta_1 = C^i C_i, \quad \Delta_2 = A^i C_i. \tag{5.6}$$

In the next section, we will use these invariants to classify the elements of $\mathcal{K}^1(\mathbb{E}^3)$ and define translational and rotational Killing vectors in terms of Δ_1 and Δ_2.

5.2 The space of translational Killing tensors

In section 4 we defined the space of translational Killing tensors $\mathcal{K}^2_T(\mathbb{E}^3)$ to the set of Killing tensors of the form (4.30). We pointed out that the upper 2×2 block of (4.30) is the general Killing tensor in the space $\mathcal{K}^2(\mathbb{E}^2)$. The fundamental $I(\mathbb{E}^2)$-invariants of this vector space are known (see, for example, [15]), and hence are also $I(\mathbb{E}^2)$-invariants of $\mathcal{K}^2_T(\mathbb{E}^3)$. It follows that $\mathcal{K}^2_T(\mathbb{E}^3)$ admits two fundamental $I(\mathbb{E}^2)$-invariants given by[8]

$$\Delta_1 = c_3, \quad \Delta_2 = [{b_{13}}^2 - {b_{23}}^2 + c_3(a_2 - a_1)]^2 + 4(b_{13}b_{23} - \alpha_3 c_3)^2 \tag{5.7}$$

(see proposition 4.1 in [15]).

5.3 The space of rotational Killing tensors

The set of all Killing tensors of the form (4.31) defines the space of rotational Killing tensors $\mathcal{K}^2_R(\mathbb{E}^3)$. This subspace of $\mathcal{K}^2(\mathbb{E}^3)$ is mapped to itself under the action of the isometry group $I(\mathbb{R})$, the group of translations about the z-axis. Trivially, the Lie algebra $i(\mathbb{R})$ is generated by the Killing vector field $\boldsymbol{X}_3$ and hence the corresponding infinitesimal generator in the parameter space is the generator $\boldsymbol{U}_3$ in (3.16) restricted to $\mathcal{K}^2_R(\mathbb{E}^3)$ which reads

$$\boldsymbol{U}_3 = -2b_{12}\frac{\partial}{\partial a_1} - c_2\frac{\partial}{\partial b_{12}}. \tag{5.8}$$

Solving the PDE $\boldsymbol{U}_3(F) = 0$ by the method of characteristics, we obtain the four fundamental $I(\mathbb{R})$-invariants of $\mathcal{K}^2_R(\mathbb{E}^3)$, namely

$$\Delta_1 = c_2, \quad \Delta_2 = {b_{12}}^2 + c_2(a_3 - a_1), \quad \Delta_3 = a_3, \quad \Delta_4 = c_3. \tag{5.9}$$

5.4 The space of Killing tensors

We now briefly describe how to derive a complete set of invariants for the full vector space $\mathcal{K}^2(\mathbb{E}^3)$ of valence-two Killing tensors in Euclidean space. As mentioned at the end of section 3, $\mathcal{K}^2(\mathbb{E}^3)$ admits fifteen fundamental $I(\mathbb{E}^3)$-invariants which can be computed by solving the system of linear PDEs

$$\boldsymbol{U}_i(F) = 0, \quad \boldsymbol{V}_i(F) = 0, \quad i = 1, 2, 3, \tag{5.10}$$

where the generators $\boldsymbol{U}_i$ and $\boldsymbol{V}_i$ are given by (3.16) and F is an analytic function in the Killing tensor parameters.

Computationally, arriving at the general solution of (5.10) is non-trivial. In particular, we have found that the method of characteristics becomes intractable when applied to (5.10). However, we have successfully computed all fifteen invariants using the method of undetermined coefficients [6]. The simplest implementation of this method is to build monomial trial functions in the Killing tensor parameters up to a fixed degree, take a linear combination of these monomials and substitute the combination into (5.10) leading to a large (sparse) system of linear equations in the

[8] The paper [15] only treats the space of "non-trivial" Killing tensors in $\mathcal{K}^2(\mathbb{E}^2)$. The invariants (5.7) form a complete set of fundamental $I(\mathbb{E}^2)$-invariants for this vector subspace. It can be shown that $\mathcal{K}^2_T(\mathbb{E}^3)$ admits two additional fundamental $I(\mathbb{E}^2)$-invariants; we do not present them here as they play no role in classifying the elements of $\mathcal{K}^2_T(\mathbb{E}^3)$.

undetermined coefficients. This approach has two obvious disadvantages. Firstly, such an ansatz does not take advantage of the apparent structure and symmetry of (5.10). Consequently, as one must construct all monomials up to and including degree five to recover all fifteen fundamental invariants, approximately 50 000 undetermined coefficients are involved; the corresponding linear system requires almost ninety hours of CPU time to solve on a modest Sun workstation! Secondly, by virtue of the ansatz, the computed invariants are in expanded form and occupy fifteen pages of output. A more effective ansatz involves constructing scalar trial functions which are "tensorial"[9] in the A^{ij}, B^{ij} and C^{ij}. For example, trial functions which are cubic in the C^{ij} include

$$C_i{}^i C_j{}^j C_k{}^k, \quad C^{ij}C_{ij}C_k{}^k, \quad C^{ij}C_j{}^kC_{ki}, \quad \epsilon_{ikm}\epsilon_{j\ell n}C^{ij}C^{k\ell}C^{mn}.$$

Implementing the method of undetermined coefficients with this ansatz yields the following fifteen fundamental $I(\mathbb{E}^3)$-invariants of $\mathcal{K}^2(\mathbb{E}^3)$:

$$\begin{aligned}
&\Delta_1 = B_i{}^i,\ \ \Delta_2 = C_i{}^i,\ \ \Delta_3 = B^{ij}C_{ij},\ \ \Delta_4 = C^{ij}C_{ij},\ \ \Delta_5 = B^{ij}B_{ji} + A^{ij}C_{ij},\\
&\Delta_6 = B^{ij}C_j{}^kC_{ki}, \quad \Delta_7 = C^{ij}C_j{}^kC_{ki}, \quad \Delta_8 = C^{ij}[B_j{}^k(B_{ik}+2B_{ki}) + A_j{}^kC_{ki}],\\
&\Delta_9 = \epsilon_{ikm}\epsilon_{j\ell n}B^{ij}B^{k\ell}B^{mn} - 2(B_i{}^{[i}B_j{}^{j]} + A^{ij}C_{ij})B_k{}^k + 6B^{ij}A_j{}^kC_{ki},\\
&\Delta_{10} = B^{ij}(B_i{}^kC_{kj} - 2B_j{}^kC_{ki}) - (B^{ij}B_{ij} + A^{ij}C_{ij})C_k{}^k + A_i{}^iC_j{}^{[j}C_k{}^{k]},\\
&\Delta_{11} = \epsilon_{i\ell m}\epsilon_{jkp}B^{ij}B^{k\ell}C^{mn}C_n{}^p + B^{ij}[B_{ij}C^{k\ell}C_{k\ell} - C_j{}^k(C_k{}^\ell B_{i\ell} + 4C_{[k}{}^\ell B_{\ell]i})]\\
&\qquad + A^{ij}C_{ij}C_k{}^{[k}C_\ell{}^{\ell]},\\
&\Delta_{12} = A_i{}^i[(C_j{}^jC_k{}^k + 3C^{jk}C_{jk})C_\ell{}^\ell - 4C^{jk}C_k{}^\ell C_{\ell j}] - 6A^{ij}C_{ij}C^{k\ell}C_{k\ell}\\
&\qquad + 6B^{ij}\{B_{ij}C^{k\ell}C_{k\ell} - C_j{}^k[(B_{ik} - 2B_{ki})C_\ell{}^\ell + 4C_k{}^\ell B_{\ell i}]\}\\
&\qquad + 12\epsilon_{i\ell m}\epsilon_{jkp}B^{ij}B^{k\ell}C^{mn}C_n{}^p,\\
&\Delta_{13} = A^{ij}(B_{ij}C_k{}^{[k}C_\ell{}^{\ell]} + B_j{}^kC_k{}^\ell C_{\ell i} - 2C_{i(j}B_{k)}{}^kC_\ell{}^\ell)\\
&\qquad + A_i{}^iC^{jk}(B_{jk}C_\ell{}^\ell - B_k{}^\ell C_{\ell j}) - B^{ij}[B_{ij}B^{k\ell}C_{k\ell} + 2C_j{}^kB_{ki}B_\ell{}^\ell\\
&\qquad + B_j{}^kB_{ik}C_\ell{}^\ell - (B_j{}^kB_{i\ell} + B_i{}^kB_{\ell j})C_k{}^\ell],\\
&\Delta_{14} = 4A_i{}^{[i}A_j{}^{j]}C_k{}^{[k}C_\ell{}^{\ell]} + 8A^{ij}(A_j{}^kC_{k[i}C_{\ell]}{}^\ell + A_k{}^kC_{[j}{}^\ell C_{\ell]i}) + A^{ij}C_{ij}(A^{k\ell}C_{k\ell}\\
&\qquad + 4B_k{}^{[k}B_\ell{}^{\ell]}) + 4C^{ij}B_j{}^kA_k{}^\ell B_{i\ell} + 16A^{ij}C_j{}^kB_{[k}{}^\ell B_{\ell]i},\\
&\Delta_{15} = A^{ij}C_{ij}[(C_k{}^kC_\ell{}^\ell - 3C^{k\ell}C_{k\ell})C_m{}^m + 2C^{k\ell}C_\ell{}^mC_{mk}]\\
&\qquad - 6A^{ij}C_j{}^kC_{ki}C_\ell{}^{[\ell}C_m{}^{m]} - 12C^{ij}B_j{}^k(C_k{}^\ell B_{i[\ell}C_{m]}{}^m + 2B_k{}^\ell C_{i[\ell}C_{m]}{}^m).
\end{aligned} \tag{5.11}$$

We have hence proven the following theorem.

Theorem 5.1. *Consider the vector space $\mathcal{K}^2(\mathbb{E}^3)$. Any algebraic $I(\mathbb{E}^3)$-invariant $\mathcal{I}$ defined over $\mathcal{K}^2(\mathbb{E}^3)$ in terms of the Killing tensor parameters (3.9) where the isometry group $I(\mathbb{E}^3)$ acts freely and regularly with six-dimensional orbits can be locally uniquely expressed as an analytic function $\mathcal{I} = F(\Delta_1, \ldots, \Delta_{15})$, where the fundamental $I(\mathbb{E}^3)$-invariants Δ_i, $i = 1, \ldots, 15$, are given by (5.11).*

We close this section by addressing a minor computational issue. It is clear from (3.7) that the Killing tensor parameters b_{11}, b_{22} and b_{33} are *not* uniquely

[9] As the A^{ij} and B^{ij} do not transform as tensors (see (3.13)), there is no reason why this ansatz should work.

Table 2: Invariant classification of Killing vectors in Euclidean space.

Classification	Invariants	
translational	$\Delta_1 = 0,$	$\Delta_2 = 0$
rotational	$\Delta_1 \neq 0,$	$\Delta_2 = 0$
helicoidal	$\Delta_1 \neq 0,$	$\Delta_2 \neq 0$

determined for a given Killing tensor $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^3)$. Thus, how does one evaluate the invariants (5.11) on a given Killing tensor? Indeed, we require an invariant method for solving the equations (3.8) for the b_{ii} in terms of the known β_i. This problem is easily rectified upon observing that the equations (3.8) in conjunction with the condition $\Delta_1 = 0 \Leftrightarrow b_{11} + b_{22} + b_{33} = 0$ yields a unique solution for the b_{ii} given by

$$b_{11} = \tfrac{1}{3}(\beta_3 - \beta_2), \quad b_{22} = \tfrac{1}{3}(\beta_1 - \beta_3), \quad b_{33} = \tfrac{1}{3}(\beta_2 - \beta_1). \tag{5.12}$$

In what follows, we shall extract the parameters b_{11}, b_{22} and b_{33} from a given $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^3)$ using (5.12).

6 Invariant classification of orthogonal coordinate webs in Euclidean space

In order to build a classification scheme for the orthogonal coordinate webs in Euclidean space based on the set of Killing tensors in $\mathcal{K}^2(\mathbb{E}^3)$ with normal eigenvectors and distinct eigenvalues, one must first know how to classify elements in the vector spaces $\mathcal{K}^1(\mathbb{E}^3)$, $\mathcal{K}^2_T(\mathbb{E}^3)$ and $\mathcal{K}^2_R(\mathbb{E}^3)$. The classification of elements in these spaces are treated in subsections 6.1, 6.2 and 6.3, respectively. We then use these results to classify the orthogonal coordinate webs of $\mathcal{K}^2(\mathbb{E}^3)$ in subsection 6.4.

6.1 Classification of Killing vectors

We shall classify the elements of $\mathcal{K}^1(\mathbb{E}^3)$ according to whether the fundamental invariants (5.6) are zero or non-zero. There are three cases to consider each of which gives rise to a canonical Killing vector. The classification scheme is summarized in table 2

1. $\Delta_1 = 0$, $\Delta_2 = 0$. In this case, the Killing vector $\boldsymbol{V}$ is of the form

$$\boldsymbol{V} = A^i \boldsymbol{X}_i, \tag{6.1}$$

where it is assumed that the A^i are not all zero so that $\boldsymbol{V}$ is non-trivial. It follows that we can use the isometry group $I(\mathbb{E}^3)$ to transform (6.1) to

$$\tilde{\boldsymbol{V}} = \tilde{a}_3 \tilde{\boldsymbol{X}}_3, \tag{6.2}$$

for some $\tilde{a}_3 \neq 0$. Indeed, the transformation rules (5.4) reduce to $\tilde{C}^i = 0$ and $\tilde{A}^i = A^j \lambda^i{}_j$. Without loss of generality, we can set the components of the translation $\delta^i = 0$. The components of the rotation $\lambda_j{}^i$ can be computed by setting $\lambda^3{}_j = (A^k A_k)^{-1/2} A_j$ and then obtaining $\lambda^1{}_j$ and $\lambda^2{}_j$ by extending

Table 3: Invariant classification of translational Killing tensors in Euclidean space.

Orthogonal coordinate web	Invariants
Cartesian	$\Delta_1 = 0, \quad \Delta_2 = 0$
circular cylindrical	$\Delta_1 \neq 0, \quad \Delta_2 = 0$
parabolic cylindrical	$\Delta_1 = 0, \quad \Delta_2 \neq 0$
elliptic-hyperbolic	$\Delta_1 \neq 0, \quad \Delta_2 \neq 0$

the vector ${\lambda^3}_j$ to a proper orthonormal basis in $\mathbb{E}^3$ (using the Gram-Schmidt algorithm or QR decomposition, for example). We observe that (6.2) defines a translation, hence we say that a Killing vector $\boldsymbol{V} \in \mathcal{K}^1(\mathbb{E}^3)$ is *translational* iff $\Delta_1 = \Delta_2 = 0$.

2. $\Delta_1 \neq 0$, $\Delta_2 = 0$. Using the isometry group $I(\mathbb{E}^3)$, we claim that any such Killing vector $\boldsymbol{V}$ can be transformed to

$$\tilde{\boldsymbol{V}} = \tilde{c}_3 \tilde{\boldsymbol{R}}_3, \tag{6.3}$$

for some $\tilde{c}_3 \neq 0$. Indeed, we can first make a translation $x^i = \hat{x}^i + \delta^i$ so that $\hat{A}^i = 0$ in the new coordinates $\hat{x}^i$. It follows from the transformation rules (5.4) and (3.12) that the δ^i must satisfy the system of linear equations ${\epsilon^i}_{jk} C^j \delta^k = A^i$, which has a solution iff $\Delta_2 = 0$. In the new coordinates $\hat{x}^i$, it follows that $\hat{\boldsymbol{V}} = C^i \hat{\boldsymbol{R}}_i$. By a similar argument to that used in the previous case, we can find a rotation ${\lambda_j}^i$ such the coordinate transformation $\hat{x}^i = {\lambda_j}^i \tilde{x}^j$ puts $\hat{\boldsymbol{V}}$ into the form (6.3). We observe that (6.3) defines a rotation, hence we say that a Killing vector $\boldsymbol{V} \in \mathcal{K}^1(\mathbb{E}^3)$ is *rotational* iff $\Delta_1 \neq 0$ and $\Delta_2 = 0$.

3. $\Delta_1 \neq 0$, $\Delta_2 \neq 0$. By similar arguments to those used in the previous two cases, it can be shown in this case that there exists a coordinate transformation (3.10) such that

$$\tilde{\boldsymbol{V}} = \tilde{a}_3 \tilde{\boldsymbol{X}}_3 + \tilde{c}_3 \tilde{\boldsymbol{R}}_3, \tag{6.4}$$

for some non-zero $\tilde{a}_3$ and $\tilde{c}_3$. The integral curves of such a Killing vector field are helices and so we say that a Killing vector $\boldsymbol{V} \in \mathcal{K}^1(\mathbb{E}^3)$ is *helicoidal* iff both $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$. We remark that we can further refine this classification into left and right-handed helicoidal Killing vectors according to the sign of Δ_2.

6.2 The space of translational Killing tensors

A classification scheme for the vector space of translational Killing tensors $\mathcal{K}^2_T(\mathbb{E}^3)$ based on the fundamental $I(\mathbb{E}^2)$-invariants (5.7) is provided in [15]. For completeness, we summarize this scheme in table 3.

6.3 The space of rotational Killing tensors

Evaluating the $I(\mathbb{R})$-invariants (5.9) on each of the rotational CKTs (4.22)–(4.25) produces a classification scheme for the vector space $\mathcal{K}^2_R(\mathbb{E}^3)$. It turns out that we

Table 4: Invariant classification of rotational Killing tensors in Euclidean space.

Orthogonal coordinate web	Invariants
circular cylindrical	$\Delta_1 = 0, \quad \Delta_2 = 0$
spherical	$\Delta_1 \neq 0, \quad \Delta_2 = 0$
prolate spheroidal	$\Delta_1 \neq 0, \quad \Delta_2 > 0$
oblate spheroidal	$\Delta_1 \neq 0, \quad \Delta_2 < 0$
parabolic	$\Delta_1 = 0, \quad \Delta_2 \neq 0$

only need to use the invariants Δ_1 and Δ_2 in (5.9) to obtain a classification. As we pointed out in section 4, we can also include the circular cylindrical web in this classification. Our classification scheme is detailed in table 4.

6.4 The space of Killing tensors

The motivation for constructing invariant classification schemes in the vector spaces treated in the previous three subsections is due to the fact that we have been unable to obtain a scheme based solely on the fifteen fundamental $I(\mathbb{E}^3)$-invariants of the full space $\mathcal{K}^2(\mathbb{E}^3)$ presented in (5.11). This is primarily because these invariants fail to discriminate amongst some of the canonical CKTs (4.18)–(4.28).

To begin, let $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^3)$ be the given CKT for which we wish to classify. As we showed in section 4, if $\boldsymbol{K}$ is constant, then it necessarily characterizes a Cartesian web. Let us therefore assume for the remainder of this section that $\boldsymbol{K}$ is not constant. The classification of $\boldsymbol{K}$ involves two main steps:

1. Determine whether $\boldsymbol{K}$ characterizes a translational, rotational or asymmetric web (according to the type of Killing vector it admits).

2. Use the classification schemes in tables 3 and 4 if $\boldsymbol{K}$ is translational or rotational, or, the classification scheme outlined in table 5 if $\boldsymbol{K}$ characterizes an asymmetric web (i.e. $\boldsymbol{K}$ admits no Killing vector).

To proceed with the first step, we let $\boldsymbol{V}$ be the general Killing vector from (5.3) and impose the condition

$$\mathcal{L}_{\boldsymbol{V}}\boldsymbol{K} = 0. \tag{6.5}$$

Equation (6.5) results in a linear system of equations in the six Killing vector parameters (5.2) which can be readily solved. It follows that the general solution of (6.5) can be decomposed as

$$\boldsymbol{V} = \ell_1\boldsymbol{V}_1 + \cdots \ell_n\boldsymbol{V}_n,$$

for some $n \leqslant 6$, where ℓ_i, $i = 1, \ldots, n$, are arbitrary non-zero constants and $\{\boldsymbol{V}_1, \ldots, \boldsymbol{V}_n\}$ is a linearly independent set of Killing vectors. If $n = 0$, we conclude that $\boldsymbol{K}$ does not admit a Killing vector, and hence characterizes an asymmetric web. Otherwise, using table 2, we classify each of the $\boldsymbol{V}_i$ according to whether they are translational, rotational, or helicoidal. Therefore, if one of the $\boldsymbol{V}_i$ is translational,

then $\boldsymbol{K}$ characterizes a translational web, otherwise $\boldsymbol{K}$ characterizes a rotational web[10].

We have now shown how to determine if $\boldsymbol{K}$ characterizes a translational, rotational or asymmetric web. To proceed, suppose that $\boldsymbol{K}$ is a translational (rotational) Killing tensor and let $\boldsymbol{V}$ be its corresponding translational (rotational) Killing vector. From the results of subsection 6.2 (6.3), we can use the isometry group $I(\mathbb{E}^3)$ to bring $\boldsymbol{V}$ to the canonical form $\tilde{a}_3\tilde{\boldsymbol{X}}_3$ ($\tilde{c}_3\tilde{\boldsymbol{R}}_3$). Applying the corresponding coordinate transformation to the Killing tensor $\boldsymbol{K}$ places it in the subspace $\mathcal{K}^2_T(\mathbb{E}^3)$ ($\mathcal{K}^2_R(\mathbb{E}^3)$). Finally, we can classify the transformed $\boldsymbol{K}$ using table 3 (4).

Suppose now that $\boldsymbol{K}$ characterizes an asymmetric web. Using the fundamental $I(\mathbb{E}^3)$-invariants (5.11) of $\mathcal{K}^2(\mathbb{E}^3)$, we shall derive a scheme for classifying $\boldsymbol{K}$. To begin, we evaluate the invariants Δ_2, Δ_4 and Δ_7 (see (5.11)) on the three asymmetric CKTs (4.26)–(4.28). It follows that for the conical and ellipsoidal tensors, $(\Delta_2, \Delta_4, \Delta_7) = (c_1+c_2+c_3, {c_1}^2+{c_2}^2+{c_3}^2, {c_1}^3+{c_2}^3+{c_3}^3)$, while for the paraboloidal tensor, $(\Delta_2, \Delta_4, \Delta_7) = (c_3, {c_3}^2, {c_3}^3)$. This motivates defining two auxiliary invariants

$$\Xi_1 = {\Delta_2}^2 - \Delta_4, \quad \Xi_2 = {\Delta_2}^3 - \Delta_7, \tag{6.6}$$

noting that $\Xi_1 = \Xi_2 = 0$ on the paraboloidal tensor (4.27). We claim that the vanishing of Ξ_1 and Ξ_2 is also a sufficient condition for $\boldsymbol{K}$ to characterize a paraboloidal web. Indeed, it follows that $\Xi_1 = \Xi_2 = 0$ in one or more of the following three cases

$$c_1 = c_2 = 0, \quad c_2 = c_3 = 0, \quad c_3 = c_1 = 0.$$

The conical case (4.26) and the ellipsoidal case (4.28) cannot have $c_1 = c_2 = 0$, since this condition reduces them to the spherical and elliptic-hyperbolic tensors, respectively. Moreover, by a rotation, the two other cases are also impossible for conical and ellipsoidal tensors. Therefore, we conclude that $\boldsymbol{K}$ characterizes a paraboloidal web if and only if $\Xi_1 = \Xi_2 = 0$.

Suppose now that $\boldsymbol{K}$ does not characterize a paraboloidal web for the remainder of this section. It is convenient to define

$$\Xi_3 = 3\Delta_4 - {\Delta_2}^2, \tag{6.7}$$

noting that $\Xi_3 = (c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_3 - c_1)^2$ on both the conical and ellipsoidal CKTs (4.26) and (4.28). Indeed, if $\Xi_3 = 0$, the web cannot possibly be a conical tensor since $c_1 = c_2 = c_3$ reduces it to a special case of the spherical tensor. Therefore, if $\Xi_3 = 0$, then $\boldsymbol{K}$ necessarily characterizes an ellipsoidal web.

To distinguish between a conical and ellipsoidal tensor in the case when $\Xi_3 \neq 0$, we define three additional auxiliary invariants given by

$$\begin{aligned} \Xi_4 &= \Delta_2\Delta_5 - 3\Delta_8 - 2\Delta_{10}, \\ \Xi_5 &= \Delta_2\Delta_{10} + \Delta_4\Delta_5 - \Delta_{11}, \\ \Xi_6 &= \Delta_2[2\Delta_2(10\Delta_2\Delta_5 + 24\Delta_8 - 3\Delta_{10}) - 72\Delta_{11} + \Delta_{12}] \\ &\quad - 48\Delta_4\Delta_8 - 20\Delta_5\Delta_7 + 16\Delta_{15} \end{aligned} \tag{6.8}$$

[10]It is impossible for all of the $\boldsymbol{V}_i$ to be helicoidal. Although the circular cylindrical web is the only coordinate web which admits a helicoidal Killing vector, it also admits both translational and rotational Killing vectors. Clearly, if one of the $\boldsymbol{V}_i$ is helicoidal, we can conclude immediately that $\boldsymbol{K}$ characterizes a circular cylindrical web.

Table 5: Invariant classification of asymmetric Killing tensors in Euclidean space.

Orthogonal coordinate web	Invariants
paraboloidal	$(\Xi_1, \Xi_2) = (0, 0)$
ellipsoidal	$(\Xi_1, \Xi_2) \neq (0, 0), \quad \Xi_3 = 0 \quad$ or
	$(\Xi_1, \Xi_2) \neq (0, 0), \quad \Xi_3 \neq 0, \quad (\Xi_4, \Xi_5, \Xi_6) \neq (0, 0, 0)$
conical	$(\Xi_1, \Xi_2) \neq (0, 0), \quad \Xi_3 \neq 0, \quad (\Xi_4, \Xi_5, \Xi_6) = (0, 0, 0)$

It follows that the invariants (6.8) all evaluate to zero on the conical tensor (4.26). We claim that $\Xi_4 = \Xi_5 = \Xi_6 = 0$ (in conjunction with $\Xi_3 \neq 0$) is also a sufficient condition for a conical tensor. Indeed, arguing by contradiction, it follows that

$$
\begin{aligned}
\Xi_4 &= (a_1 + a_2 - 2a_3)c_1c_2 + (a_2 + a_3 - 2a_1)c_2c_3 + (a_3 + a_1 - 2a_2)c_3c_1, \\
\Xi_5 &= (c_1c_2 + c_2c_3 + c_3c_1)[(a_1 + a_2 - 2a_3)c_3 + (a_2 + a_3 - 2a_1)c_1 \\
&\qquad + (a_3 + a_1 - 2a_2)c_2], \\
\Xi_6 &= 12c_1c_2c_3[(2a_1 - a_2 - a_3)c_1 + (2a_2 - a_3 - a_1)c_2 + (2a_3 - a_1 - a_2)c_3],
\end{aligned}
$$

on the ellipsoidal tensor (4.28) and that $\Xi_4 = \Xi_5 = \Xi_6 = 0$ in one or more of the following five cases[11]:

$$c_1 = c_2 = 0, \quad c_2 = c_3 = 0, \quad c_3 = c_1 = 0, \quad c_1 = c_2 = c_3, \quad a_1 = a_2 = a_3.$$

By previous arguments, the first three cases are impossible if the tensor is ellipsoidal. The fourth case can also be eliminated since $\Xi_3 \neq 0$. Finally, the fifth case is impossible since it reduces to a conical tensor. Therefore, we conclude that if $\Xi_3 \neq 0$, then the tensor is conical if and only if $\Xi_4 = \Xi_5 = \Xi_6 = 0$.

This completes the derivation of the classification scheme for the set of asymmetric CKTs in Euclidean space. These results are summarized in table 5.

7 Transformations to canonical form

Once a CKT in Euclidean space has been classified using the scheme detailed in the previous section, the isometry group $I(\mathbb{E}^3)$ can be used to transform the Killing tensor into its corresponding canonical form. This step leads directly to the transformation to separable coordinates. In this section we provide methods for determining the transformations to canonical form. As we shall see, the majority of the calculations amount to elementary linear algebra.

The procedure can be summarized as follows. Suppose that K^{ij} are the components of a CKT with respect to Cartesian coordinates x^i. Under the action of $I(\mathbb{E}^3)$, the transformation from the original set of Cartesian coordinates x^i to another set $\tilde{x}^i$ is given by (3.10), i.e.

$$x^i = \lambda_j{}^i \tilde{x}^j + \delta^i. \tag{7.1}$$

Thus, we need to determine the rotation $\lambda_j{}^i \in SO(3)$ and translation $\delta^i \in \mathbb{R}^3$ which brings K^{ij} to its appropriate canonical form $\tilde{K}^{ij}$ given by one of the eleven cases

[11] This calculation is facilitated by use of the Maple Gröbner basis package.

(4.18)-(4.28) (in the coordinates $\tilde{x}^i$). Moreover, any essential parameters appearing in the tensor also need to be determined (e.g. the parameter a appearing in the elliptic-hyperbolic tensor listed in (4.10)). Once $\lambda_j{}^i$, δ^i and all essential parameters are known, the transformation from Cartesian coordinates x^i to separable coordinates u^i is

$$x^i = \lambda_j{}^i T^j(u^k) + \delta^i, \tag{7.2}$$

where $x^i = T^i(u^j)$ is the standard coordinate transformation associated with the separable coordinates tabulated in (4.7)–(4.17).

To carry out this procedure, we use the transformation rules relating the parameter matrices of K^{ij} to those of its canonical form $\tilde{K}^{ij}$ (see (3.6)). In matrix form, these transformation rules read

$$\tilde{\boldsymbol{A}} = \boldsymbol{\lambda}^t \boldsymbol{A} \boldsymbol{\lambda} + 2\,\mathcal{S}(\boldsymbol{\lambda}^t \boldsymbol{B} \boldsymbol{\mu}) + \boldsymbol{\mu}^t \boldsymbol{C} \boldsymbol{\mu}, \tag{7.3a}$$
$$\tilde{\boldsymbol{B}} = \boldsymbol{\lambda}^t \boldsymbol{B} \boldsymbol{\lambda} + \boldsymbol{\mu}^t \boldsymbol{C} \boldsymbol{\lambda}, \tag{7.3b}$$
$$\tilde{\boldsymbol{C}} = \boldsymbol{\lambda}^t \boldsymbol{C} \boldsymbol{\lambda}, \tag{7.3c}$$

where $\boldsymbol{x} = \boldsymbol{\lambda}\tilde{\boldsymbol{x}} + \boldsymbol{\delta}$, $\mathcal{S}$ denotes the symmetric part and

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i = x^i, \quad \tilde{\boldsymbol{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}_i = \tilde{x}^i, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1{}^1 & \lambda_2{}^1 & \lambda_3{}^1 \\ \lambda_1{}^2 & \lambda_2{}^2 & \lambda_3{}^2 \\ \lambda_1{}^3 & \lambda_2{}^3 & \lambda_3{}^3 \end{pmatrix}_{ij} = \lambda_j{}^i,$$

$$\boldsymbol{\delta} = \begin{pmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \end{pmatrix}_i = \delta^i, \;\; \boldsymbol{\mu} = \begin{pmatrix} \lambda_1{}^2\delta^3 - \lambda_1{}^3\delta^2 & \lambda_2{}^2\delta^3 - \lambda_2{}^3\delta^2 & \lambda_3{}^2\delta^3 - \lambda_3{}^3\delta^2 \\ \lambda_1{}^3\delta^1 - \lambda_1{}^1\delta^3 & \lambda_2{}^3\delta^1 - \lambda_2{}^1\delta^3 & \lambda_3{}^3\delta^1 - \lambda_3{}^3\delta^1 \\ \lambda_1{}^1\delta^2 - \lambda_1{}^2\delta^1 & \lambda_2{}^1\delta^2 - \lambda_2{}^2\delta^1 & \lambda_3{}^1\delta^2 - \lambda_3{}^2\delta^1 \end{pmatrix}_{ij} = \mu_j{}^i.$$

The identities

$$\boldsymbol{\mu}\boldsymbol{\lambda}^t = \begin{pmatrix} 0 & \delta^3 & -\delta^2 \\ -\delta^3 & 0 & \delta^1 \\ \delta^2 & -\delta^1 & 0 \end{pmatrix}, \;\; \boldsymbol{\mu}\boldsymbol{\mu}^t = \begin{pmatrix} (\delta^2)^2 + (\delta^3)^2 & -\delta^1\delta^2 & -\delta^3\delta^1 \\ -\delta^1\delta^2 & (\delta^3)^2 + (\delta^1)^2 & -\delta^2\delta^3 \\ -\delta^3\delta^1 & -\delta^2\delta^3 & (\delta^1)^2 + (\delta^2)^2 \end{pmatrix}, \tag{7.4}$$

shall prove useful and, in addition, the inverse of (7.3) which reads

$$\boldsymbol{A} = \boldsymbol{\lambda}\tilde{\boldsymbol{A}}\boldsymbol{\lambda}^t + 2\,\mathcal{S}(\boldsymbol{\lambda}\tilde{\boldsymbol{B}}\boldsymbol{\mu}^t) + \boldsymbol{\mu}\tilde{\boldsymbol{C}}\boldsymbol{\mu}^t, \tag{7.5a}$$
$$\boldsymbol{B} = \boldsymbol{\lambda}\tilde{\boldsymbol{B}}\boldsymbol{\lambda}^t + \boldsymbol{\mu}\tilde{\boldsymbol{C}}\boldsymbol{\lambda}^t, \tag{7.5b}$$
$$\boldsymbol{C} = \boldsymbol{\lambda}\tilde{\boldsymbol{C}}\boldsymbol{\lambda}^t. \tag{7.5c}$$

Our procedure for determining the transformation to canonical form for the cases of translational and rotational CKTs is provided in subsections 7.1 and 7.2, respectively. The three asymmetric CKTs are each treated separately in subsections 7.3–7.5.

7.1 The translational Killing tensors

Let us consider first the Cartesian CKT. Since it is necessarily a constant tensor, the transformation rules (7.3) reduce to $\tilde{\boldsymbol{A}} = \boldsymbol{\lambda}^t\boldsymbol{A}\boldsymbol{\lambda}$, where $\tilde{\boldsymbol{A}} = \mathrm{diag}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$, on account of (4.18). Trivially, the $\tilde{a}_i$ are the eigenvalues of $\boldsymbol{A}$, the columns of $\boldsymbol{\lambda}$ are the (normalized) eigenvectors of $\boldsymbol{A}$ and the translation $\boldsymbol{\delta}$ is arbitrary.

Suppose now that the CKT is circular cylindrical, parabolic cylindrical or elliptic-hyperbolic. We may assume without loss of generality that the tensor has the form (4.30), since it must necessarily be of this form in order to carry out the classification scheme in section 6. Consequently, the rotation and translation are of the form

$$\boldsymbol{\lambda} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\delta} = \begin{pmatrix} \delta^1 \\ \delta^2 \\ 0 \end{pmatrix},$$

and thus the problem reduces to finding ϕ, δ^1 and δ^2. The derivation of these parameters for each of the three translational CKTs under consideration is provided in [15] (see table 1, p 1432). We now restate these results in our notation.

1. Circular cylindrical case: The rotation angle ϕ is arbitrary and

$$\delta^1 = \frac{b_{23}}{c_3}, \quad \delta^2 = -\frac{b_{13}}{c_3}.$$

2. Parabolic cylindrical case: If $b_{23} \neq 0$, then

$$\tan\phi = -\frac{b_{13}}{b_{23}},$$

and $\phi = \frac{\pi}{2}$ for $b_{23} = 0$ (unique mod π). The components of the translation are

$$\delta^1 = \frac{b_{23}(a_2 - a_1) + 2\alpha_3 b_{13}}{2({b_{13}}^2 + {b_{23}}^2)}, \quad \delta^2 = \frac{b_{13}(a_2 - a_1) - 2\alpha_3 b_{23}}{2({b_{13}}^2 + {b_{23}}^2)}.$$

3. Elliptic-hyperbolic case: Let

$$\sigma_1 = {b_{13}}^2 - {b_{23}}^2 + c_3(a_2 - a_1), \quad \sigma_2 = \alpha_3 c_3 - b_{13} b_{23}, \quad \Delta = {\sigma_1}^2 + 4{\sigma_2}^2$$

(and note that Δ is one of the fundamental invariants (5.7)). Then,

$$\tan\phi = \begin{cases} 0, & \text{if} \quad \sigma_2 = 0 \text{ and } \sigma_1 < 0, \\ \infty, & \text{if} \quad \sigma_2 = 0 \text{ and } \sigma_1 > 0, \\ \frac{\sigma_1 + \sqrt{\Delta}}{2\sigma_2}, & \text{if} \quad \sigma_2 \neq 0, \end{cases}$$

(ϕ unique mod π), and δ^1 and δ^2 are the same as in the circular cylindrical case. Moreover, the essential parameter a satisfies

$$a^2 = \frac{\tilde{a}_1 - \tilde{a}_2}{\tilde{c}_3} = \frac{\sqrt{\Delta}}{{c_3}^2}.$$

7.2 The rotational Killing tensors

By the same reasoning used in the previous subsection, we may assume without loss of generality that the rotational Killing tensor has the form (4.31). As the isometry group for this subspace of rotational webs is the group of translations about the z-axis, we set ${\lambda_j}^i = {\delta_j}^i$ and $\delta^1 = \delta^2 = 0$ in the transformation rules (7.3). The

determination of δ^3 and hence the transformation to canonical form thus becomes a trivial calculation. It follows that

$$\delta^3 = \frac{b_{12}}{c_2}$$

for the spherical, prolate spheroidal and oblate spheroidal cases and

$$\delta^3 = \frac{a_1 - a_3}{2b_{12}}$$

for the parabolic case. Finally, the essential parameter a appearing in the transformation from Cartesian to prolate (oblate) spheroidal coordinates satisfies

$$a^2 = \pm\frac{\tilde{a}_3 - \tilde{a}_1}{\tilde{c}_2} = \pm\frac{\Delta_2}{{\Delta_1}^2},$$

where Δ_1 and Δ_2 are the fundamental $I(\mathbb{R})$-invariants (5.9) and the positive (negative) signs correspond to the prolate (oblate) spheroidal tensor.

7.3 The conical case

The parameter matrices associated with the canonical form of the conical Killing tensor specialize to

$$\tilde{\boldsymbol{A}} = \tilde{a}_1\mathbf{1}, \quad \tilde{\boldsymbol{B}} = \mathbf{0}, \quad \tilde{\boldsymbol{C}} = \mathrm{diag}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$$

(see (4.26)). From (7.3c), $\tilde{\boldsymbol{C}} = \boldsymbol{\lambda}^t\boldsymbol{C}\boldsymbol{\lambda}$, hence, as in the Cartesian case, the $\tilde{c}_i$ are the eigenvalues of $\boldsymbol{C}$ and the columns of $\boldsymbol{\lambda}$ are the (normalized) eigenvectors of $\boldsymbol{C}$. It follows that the essential parameters b and c satisfy

$$\frac{b^2}{c^2} = \frac{\tilde{c}_2 - \tilde{c}_1}{\tilde{c}_3 - \tilde{c}_1},$$

thus, in order to satisfy the condition $b^2 < c^2$, we can order the eigenvalues such that $\tilde{c}_1 < \tilde{c}_2 < \tilde{c}_3$. Finally, substituting (7.3c) into (7.5b) leads to $\boldsymbol{B} = \boldsymbol{\mu}\boldsymbol{\lambda}^t\boldsymbol{C}$, which can easily be solved for $\boldsymbol{\delta}$ noting the identity (7.4).

7.4 The paraboloidal case

The parameter matrices associated with the canonical form of the paraboloidal Killing tensor specialize to

$$\tilde{\boldsymbol{A}} = \mathrm{diag}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3), \quad \tilde{\boldsymbol{B}} = \begin{pmatrix} 0 & \tilde{b}_{12} & 0 \\ \tilde{b}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\boldsymbol{C}} = \mathrm{diag}(0, 0, \tilde{c}_3), \tag{7.6}$$

subject to the constraint

$$\tilde{b}_{12}[\tilde{b}_{12}\tilde{b}_{21} + \tilde{c}_3(\tilde{a}_2 - \tilde{a}_3)] + \tilde{b}_{21}[\tilde{b}_{12}\tilde{b}_{21} + \tilde{c}_3(\tilde{a}_1 - \tilde{a}_3)] = 0 \tag{7.7}$$

(see (4.27)). It follows that the essential constants b and c satisfy

$$\begin{cases} c = \frac{\tilde{a}_2 - \tilde{a}_3}{2\tilde{b}_{12}}, & c - b = \frac{\tilde{a}_2 - \tilde{a}_1}{2\tilde{b}_{12}}, \quad \text{if} \quad \tilde{c}_3 = 0, \\ b = \frac{\tilde{a}_1 - \tilde{a}_2}{2\tilde{b}_{12}}, & c - b = \frac{\tilde{b}_{12}}{2\tilde{c}_3}, \quad \text{if} \quad \tilde{c}_3 \neq 0, \ \tilde{b}_{21} = 0, \\ c = \frac{\tilde{a}_1 - \tilde{a}_2}{2\tilde{b}_{21}}, & c - b = \frac{\tilde{b}_{21}}{2\tilde{c}_3}, \quad \text{if} \quad \tilde{c}_3 \neq 0, \ \tilde{b}_{12} = 0, \\ b = \frac{\tilde{a}_1 - \tilde{a}_3}{2\tilde{b}_{12}}, & c - b = \frac{\tilde{b}_{12} + \tilde{b}_{21}}{2\tilde{c}_3}, \quad \text{if} \quad \tilde{c}_3 \neq 0, \ \tilde{b}_{12} \neq 0, \ \tilde{b}_{21} \neq 0, \end{cases} \tag{7.8}$$

together with the condition $b > c$. From (7.3c), $\tilde{\boldsymbol{C}} = \boldsymbol{\lambda}^t \boldsymbol{C} \boldsymbol{\lambda}$, hence it follows from (7.6) that $\boldsymbol{C}$ necessarily has a zero eigenvalue of multiplicity two and one other eigenvalue $\tilde{c}_3$. We now consider the two cases $\tilde{c}_3 = 0$ and $\tilde{c}_3 \neq 0$ separately.

If $\tilde{c}_3 = 0$, then it follows from (7.7) that $\tilde{b}_{21} = -\tilde{b}_{12}$. Moreover, (7.3c) is trivially satisfied and (7.3b) reduces to $\tilde{\boldsymbol{B}} = \boldsymbol{\lambda}^t \boldsymbol{B} \boldsymbol{\lambda}$. This implies that $\tilde{\boldsymbol{B}}^2 = \boldsymbol{\lambda}^t \boldsymbol{B}^2 \boldsymbol{\lambda}$, where $\tilde{\boldsymbol{B}}^2 = \operatorname{diag}(-\tilde{b}_{12}{}^2, -\tilde{b}_{12}{}^2, 0)$. Therefore, the negative eigenvalue of $\boldsymbol{B}^2$ determines $\tilde{b}_{12}$; we can take its sign to be positive without loss of generality. The normalized eigenvectors of $\boldsymbol{B}^2$ determine $\boldsymbol{\lambda}$ up to a rotation in the eigenspace associated with the negative eigenvalue which fixes $\boldsymbol{\lambda}$ up to a parameter ψ. Finally, it follows that (7.5a) reduces to $\boldsymbol{A} = 2\,\mathcal{S}(\boldsymbol{B}\boldsymbol{\lambda}\boldsymbol{\mu}^t) + \boldsymbol{\lambda}\tilde{\boldsymbol{A}}\boldsymbol{\lambda}^t$. This equation can be solved for the $\tilde{a}_i$, δ^i, and ψ which, in general, yields multiple solutions; a particular solution satisfying the condition $b > c$ (see (7.8)) can be selected.

Finally, if $\tilde{c}_3 \neq 0$, then the eigenproblem (7.3c) uniquely determines $\tilde{c}_3$ and $\boldsymbol{\lambda}$ up to a parameter ψ. Equations (7.5a) and (7.5b) can then be solved for $\tilde{b}_{12}$, $\tilde{b}_{21}$, $\tilde{a}_i$, δ^i, and ψ, in conjunction with the condition $b > c$.

7.5 The ellipsoidal case

The parameter matrices associated with the canonical form of the ellipsoidal Killing tensor specialize to

$$\tilde{\boldsymbol{A}} = \operatorname{diag}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3), \quad \tilde{\boldsymbol{B}} = \boldsymbol{0}, \quad \tilde{\boldsymbol{C}} = \operatorname{diag}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3), \tag{7.9}$$

subject to the constraint

$$(\tilde{a}_1 - \tilde{a}_2)\tilde{c}_1\tilde{c}_2 + (\tilde{a}_2 - \tilde{a}_3)\tilde{c}_2\tilde{c}_3 + (\tilde{a}_3 - \tilde{a}_1)\tilde{c}_3\tilde{c}_1 = 0 \tag{7.10}$$

(see (4.28)). It follows that the essential constants a, b and c satisfy

$$\begin{cases} a - b = \frac{\tilde{a}_1 - \tilde{a}_2}{\tilde{c}_3}, & c - a = \frac{\tilde{a}_3 - \tilde{a}_1}{\tilde{c}_2}, \quad \text{if} \quad \tilde{c}_2 \neq 0,\ \tilde{c}_3 \neq 0, \\ a - b = \frac{\tilde{a}_1 - \tilde{a}_2}{\tilde{c}_3}, & c - b = \frac{\tilde{a}_1 - \tilde{a}_2}{\tilde{c}_1}, \quad \text{if} \quad \tilde{c}_2 = 0,\ \tilde{c}_3 \neq 0, \\ c - a = \frac{\tilde{a}_3 - \tilde{a}_1}{\tilde{c}_2}, & c - b = \frac{\tilde{a}_3 - \tilde{a}_1}{\tilde{c}_1}, \quad \text{if} \quad \tilde{c}_2 \neq 0,\ \tilde{c}_3 = 0, \end{cases} \tag{7.11}$$

together with the condition $a > b > c$. As in the conical case, (7.3c) implies that the $\tilde{c}_i$ are the eigenvalues of $\boldsymbol{C}$ and the columns of $\boldsymbol{\lambda}$ are the (normalized) eigenvectors of $\boldsymbol{C}$. There are two cases to consider: (1) the $\tilde{c}_i$ are all distinct and (2) the $\tilde{c}_i$ are all equal[12].

If the $\tilde{c}_i$ are all distinct, then the matrix $\boldsymbol{\lambda}$ is uniquely determined (up to the ordering of the columns) and $\boldsymbol{\delta}$ can be computed as in the conical case. Upon obtaining $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{C}}$, the condition $a > b > c$ should be verified and, if necessary, the eigenvalues $\tilde{c}_i$ may need to be reordered accordingly.

If all of the $\tilde{c}_i$ are equal, then (7.3c) is trivially satisfied. Equation (7.5b) reduces to $\boldsymbol{B} = \tilde{c}_1 \boldsymbol{\mu}\boldsymbol{\lambda}^t$ which can be solved for $\boldsymbol{\delta}$ using the identity (7.4). Finally, (7.5a) simplifies to $\boldsymbol{A} - \tilde{c}_1 \boldsymbol{\mu}\boldsymbol{\mu}^t = \boldsymbol{\lambda}\tilde{\boldsymbol{A}}\boldsymbol{\lambda}^t$. This eigenproblem can be solved to obtain $\boldsymbol{\lambda}$.

[12]The case of only two equal $\tilde{c}_i$ is impossible, for such a Killing tensor would characterize either an elliptic-hyperbolic, prolate spheroidal or oblate spheroidal web.

8 Main algorithm

The above-presented considerations lead to a systematic and computationally efficient method of determining separable coordinates for the natural Hamiltonian (4.1). We emphasize that our algorithm is *purely algebraic*, and hence is well suited for implementation in a symbolic computer algebra system. Indeed, as we mentioned in section 1, the algorithm has been fully implemented into Maple through the `KillingTensor` package. We now summarize the three main steps of the algorithm.

(1) Impose the compatibility condition. Using the given potential V in terms of Cartesian coordinates x^i and a generic Killing tensor $\boldsymbol{K}$ of the form (3.7), impose the compatibility condition (4.3) to obtain the equivalent conditions on the Killing tensor parameters (3.9). Computationally, this step amounts to solving a system of linear equations in the parameters (3.9).

(2) Extract the orthogonal coordinate webs. Decompose the general solution obtained in step (1) into the form

$$\boldsymbol{K} = \ell_0 \boldsymbol{g} + \ell_1 \boldsymbol{K}_1 + \cdots + \ell_n \boldsymbol{K}_n, \tag{8.1}$$

where ℓ_i, $i = 1, \ldots, n$ are arbitrary constants, $\boldsymbol{g}$ is the metric tensor and $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_n\}$ is a linearly independent set of Killing tensors, noting that $n \leqslant 19$ since $\dim \mathcal{K}^2(\mathbb{E}^3) = 20$. By theorem 4.1, each $\boldsymbol{K}_i$ must necessarily have normal eigenvectors and distinct eigenvalues if it is to characterize separation in one of the eleven separable coordinate systems. The former can be verified using the TSN conditions (4.4) while the latter can be verified efficiently by computing the discriminant of the characteristic polynomial of $\boldsymbol{K}_i$ and checking that it does not vanish identically. Finally, relabel the $\boldsymbol{K}_i$ so that $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_m$, $m \leqslant n$, are CKTs.

(3) Classify each Killing tensor and transform to canonical form. For each $\boldsymbol{K}_i$, $i = 1, \ldots, m$, in step (2), classify $\boldsymbol{K}_i$ using the scheme in section 6. Finally, using the techniques described in section 7, determine the transformation (7.1) which brings $\boldsymbol{K}_i$ to its appropriate canonical form. The transformation to separable coordinates can be carried out using equation (7.2).

Remark. Because of the non-linearity of the integrable eigenvector and distinct eigenvalue conditions as well as the non-linearity of the fundamental invariants, certain linear combinations of the $\boldsymbol{K}_i$, $i = 1, \ldots, n$ in step (2) may produce Killing tensors which characterize separability in coordinate systems not characterized by the $\boldsymbol{K}_i$, $i = 1, \ldots, m$. In fact, it may be possible to construct such a linear combination where the individual Killing tensors of the combination fail to have normal eigenvectors or distinct eigenvalues. This will be illustrated by the example in the next section.

9 Application: The Calogero-Moser system

We apply the algorithm of section 8 to the three-body inverse square Calogero-Moser system with equal masses. It is defined by the natural Hamiltonian (4.1) with potential

$$V = \frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2}. \tag{9.1}$$

Solving the compatibility condition (4.3) with the potential (9.1) yields

$$\boldsymbol{K} = a_1 \boldsymbol{g} + \alpha_1 \boldsymbol{K}_1 + b_{32} \boldsymbol{K}_2 + c_3 \boldsymbol{K}_3 + \gamma_3 \boldsymbol{K}_4, \tag{9.2}$$

where

$$K_1^{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad K_4^{ij} = \begin{pmatrix} -2yz & (x+y-z)z & (z+x-y)y \\ (x+y-z)z & -2zx & (z+y-x)x \\ (z+x-y)y & (z+y-x)x & -2xy \end{pmatrix},$$
$$K_2^{ij} = \begin{pmatrix} 2y+2z & -x-y & -z-x \\ -x-y & 2z+2x & -y-z \\ -z-x & -y-z & 2x+2y \end{pmatrix}, \quad K_3^{ij} = \begin{pmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & x^2+y^2 \end{pmatrix}. \tag{9.3}$$

Using (4.4), we find that $\boldsymbol{K}$ has normal eigenvectors for all a_1, α_3, b_{32}, c_3 and γ_3. However, it follows that only $\boldsymbol{K}_2$ and $\boldsymbol{K}_4$ have distinct eigenvalues. It is also useful to note that $\boldsymbol{K}$ in (9.2) admits a Killing vector

$$\boldsymbol{V} = (y-z)\boldsymbol{X}_1 + (z-x)\boldsymbol{X}_2 + (x-y)\boldsymbol{X}_3. \tag{9.4}$$

Using the Killing vector classification scheme in section (6.1), it follows from table (6.1) that $\boldsymbol{V}$ is rotational and the transformation

$$x^i = \lambda_j{}^i \tilde{x}^j + \delta^i, \quad \lambda_j{}^i = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ -1 & -\sqrt{3} & \sqrt{2} \end{pmatrix}_{ij}, \quad \delta^i = 0, \tag{9.5}$$

brings (9.4) to the canonical form (6.3). Let us now apply step (3) of the algorithm in section 8 to each of the CKTs in (9.3).

Applying the transformation (9.5) to K_2^{ij} yields

$$\tilde{K}_2^{ij} = \sqrt{3} \begin{pmatrix} 2\tilde{z} & 0 & -\tilde{x} \\ 0 & 2\tilde{z} & -\tilde{y} \\ -\tilde{x} & -\tilde{y} & 0 \end{pmatrix}.$$

It follows that $\tilde{\boldsymbol{K}}_2 \in \mathcal{K}_R^2(\mathbb{E}^3)$ with respect to the transformed Cartesian coordinates $\tilde{x}^i$, and thus, by table 4, we see that $\boldsymbol{K}_2$ characterizes a parabolic web. Finally, we observe from (4.25) that $\tilde{K}_2^{ij}$ is already in canonical form. Therefore, it follows from (9.5) together with (4.14) that the transformation to separable parabolic coordinates (μ, ν, ψ) is given by

$$\begin{aligned} x &= \tfrac{\sqrt{2}}{\sqrt{3}}\mu\nu\cos\psi + \tfrac{1}{2\sqrt{3}}(\mu^2-\nu^2), \\ y &= -\tfrac{1}{\sqrt{6}}\mu\nu\cos\psi + \tfrac{1}{\sqrt{2}}\mu\nu\sin\psi + \tfrac{1}{2\sqrt{3}}(\mu^2-\nu^2), \\ z &= -\tfrac{1}{\sqrt{6}}\mu\nu\cos\psi - \tfrac{1}{\sqrt{2}}\mu\nu\sin\psi + \tfrac{1}{2\sqrt{3}}(\mu^2-\nu^2). \end{aligned}$$

Proceeding similarly for $\boldsymbol{K}_4$, it follows that it enjoys the form

$$\tilde{K}_4^{ij} = \begin{pmatrix} 2\tilde{y}^2-\tilde{z}^2 & -2\tilde{x}\tilde{y} & \tilde{z}\tilde{x} \\ -2\tilde{x}\tilde{y} & 2\tilde{x}^2-\tilde{z}^2 & \tilde{y}\tilde{z} \\ \tilde{z}\tilde{x} & \tilde{y}\tilde{z} & -\tilde{x}^2-\tilde{y}^2 \end{pmatrix},$$

under the transformation (9.5). We conclude from table 4 that $\tilde{K}_4^{ij}$ characterizes a spherical web and is in canonical form upon comparison with (4.22). Thus, the transformation to separable spherical coordinates is given by (9.5) together with (4.11).

As mentioned in the remark at the end of section 8, we can take various linear combinations of $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_4$ in an attempt to find additional separable coordinate systems. For example, let $\boldsymbol{K}_{5,6} = \boldsymbol{K}_3 \pm \boldsymbol{K}_1$. It follows that $\boldsymbol{K}_{5,6}$ both have distinct eigenvalues (even though $\boldsymbol{K}_1$ does not). Applying the transformation (9.5) to $\boldsymbol{K}_{5,6}$ yields

$$\tilde{K}^{ij}_{5,6} = \begin{pmatrix} \mp 1 + \tilde{y}^2 + \tilde{z}^2 & -\tilde{x}\tilde{y} & -\tilde{z}\tilde{x} \\ -\tilde{x}\tilde{y} & \mp 1 + \tilde{z}^2 + \tilde{x}^2 & -\tilde{y}\tilde{z} \\ -\tilde{z}\tilde{x} & -\tilde{z}\tilde{y} & \pm 2 + \tilde{x}^2 + \tilde{y}^2 \end{pmatrix}.$$

¿From table 4 we conclude that $\boldsymbol{K}_5$ and $\boldsymbol{K}_6$ characterize the prolate spheroidal and oblate spheroidal webs, respectively. Moreover, $\tilde{K}^{ij}_{5,6}$ is in canonical form (compare with (4.24) and (4.25)) and the essential parameter appearing in the transformation from Cartesian to prolate (or oblate) spheroidal coordinates is $a = \sqrt{3}$ for both cases (see section 7.2).

Finally, it follows that $\boldsymbol{K}_7 = \boldsymbol{K}_1 + \boldsymbol{K}_3 + \boldsymbol{K}_4$ has distinct eigenvalues and admits a translational Killing vector

$$\boldsymbol{V} = \boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3. \tag{9.6}$$

It follows that the transformation (9.5) brings (9.6) to the canonical form (6.2). Applying this transformation to K^{ij}_7 yields

$$\tilde{K}^{ij}_7 = \begin{pmatrix} -1 + 3\tilde{y}^2 & -3\tilde{x}\tilde{y} & 0 \\ -3\tilde{x}\tilde{y} & -1 + 3\tilde{x}^2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

which characterizes the circular cylindrical web.

To summarize, we have shown using the algorithm of section 8 that the Calogero-Moser system with potential (9.1) separates in five orthogonally separable coordinate systems, namely, circular cylindrical, spherical, prolate spheroidal, oblate spheroidal and parabolic. This result is consistent with that found in [3] and [9].

We conclude by making two remarks. Firstly, our analysis is *exhaustive* in the sense that we have found *all* possible orthogonally separable coordinate systems for which the Calogero-Moser system separates, since it follows that the general Killing tensor (9.2) admits the rotational Killing vector (9.4) and hence can only characterize a rotational web. Secondly, the same conclusions from this section hold for a weighted Calogero-Moser system with unequal masses. More precisely, the Hamiltonian system (4.1) with potential

$$V = \frac{g_1}{(m_1 x - m_2 y)^2} + \frac{g_2}{(m_2 y - m_3 z)^2} + \frac{g_3}{(m_3 z - m_1 x)^2}$$

separates in the five aforementioned coordinate systems, where g_i and m_i are constants and $m_i > 0$. Moreover, in all five cases, the transformation to separable coordinates is given by

$$x^i = \lambda_j{}^i T^j(u^k) + \delta^i, \quad \lambda_j{}^i = \begin{pmatrix} m_1 M N^{-1} & 0 & m_2 m_3 M \\ -m_2 {m_3}^2 M N & m_2 N & m_3 m_1 M \\ -m_3 {m_2}^2 M N & -m_3 N & m_1 m_2 M \end{pmatrix}_{ij}, \quad \delta^i = 0,$$

where

$$M = ({m_1}^2 {m_2}^2 + {m_2}^2 {m_3}^2 + {m_3}^2 {m_1}^2)^{-1/2}, \quad N = ({m_2}^2 + {m_3}^2)^{-1/2}.$$

The authors believe that this result is new.

10 Conclusions

In this paper we solve a non-trivial problem of the geometry of orthogonal coordinate webs, namely the classification of the eleven orthogonal coordinate webs in $\mathbb{E}^3$ in terms of the invariants of the corresponding vector spaces of Killing two-tensors and vectors. Notably, the original solution presented here fits well the approach to geometry of Felix Klein presented in his Erlangen Program. Moreover, the results are successfully applied to the integrability problem of the Hamiltonian systems defined in $\mathbb{E}^3$. From this viewpoint, the well-known Calogero-Moser super-separable Hamiltonian system has been integrated within the framework of the (orthogonal) Hamilton-Jacobi theory of separation of variables. The other three-dimensional pseudo-Riemannian flat space that is amendable to the methods developed in this work is Minkowski space $\mathbb{E}^{2,1}$. The work in this direction is underway.

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